On Modular Curves: Scholze's Generalization of the Langlands-Kottwitz Method to Places of Bad Reduction

Jeremiah Edwards

L'Université de Pierre et Marie Curie
Paris, France

June 11, 2012

Abstract

We describe the Langlands-Kottwitz method for counting points mod p of Shimura varieties at places of good reduction in the special case of the modular curve. We then explicit the calculation of the semisimple trace of Frobenius on the nearby cycles sheaves, following the recent work of Peter Scholze. This effectively extends the Langlands-Kottwitz approach to all places.

Contents

1 Introduction

There is a classical technique, first developed by Langlands and Kottwitz, for counting the number of points on certain Shimura varieties[7] [18]. This approach has been extended by Peter Scholze, first in the case of the moduli space of certain families of elliptic curves [25], then later for more general Shimura varieties [23]. Recently, Scholze's techniques have led to a simplified proof of the local Langlands correspondence for GL_n over p-adic fields [24]. The goal of this paper is to give a concise overview of both the classical Langlands-Kottwitz theory and the more recent developments due to Scholze.

We assume little familiarity with subject matter, beginning, in Section 2, with an overview of pertinent facts about representation theory, Hecke algebras, and the Bernstein center.

Fixing a prime p and an integer $N \geq 3$ prime to p, in Section3, we define the moduli space $\mathcal{M}_N$ of elliptic curves with level-N-structure. We also develop the tools necessary for discussing the Langlands-Kottwitz theory at places of good reduction. More precisely, we count the number of points of $\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E)$, the set of elliptic curves over $\mathbb{F}_{p^\alpha}$ which are isogenous to a given curve E. This is Theorem 3.12.

In Section 4 we give an overview of several more advanced constructions, including a generalized moduli space $\mathcal{M}_{\Gamma(p^r),M}$, the nearby cycles functor $R\psi$, and the semisimple trace of Frobenius $\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_q^r \,|\cdot)$. These are the key ingredients which Scholze uses in his extention of Langlands-Kottwitz.

We then devote Section 5 to the calculation of the semisimple trace of Frobenius on the nearby cycles of the modular curve (Theorem 5.15). To do this, we make use of an explicit calculation of the nearby cycles (Theorem 4.9) and a form of the Fundamental Lemma (Theorem 5.6).

Finally, we conclude by showing how Scholze's generalization of the approach of Langlands and Kottwitz allows one to calculate the semisimple local factor of the Hasse-Weil zeta function for the modular curve $\mathcal{M}_{\Gamma(p^r),M}$. This is done at the end of Section 5.3.

Recall that for any variety X over a local field F, The Hasse-Weil local factor $\zeta(X,s)$ is an invariant associated to the cohomology of X. If F has residue field $\mathbb{F}_q$, we denote by $I_F \subset \mathrm{Gal}(\overline{F}/F)$ the standard inertia subgroup. We then have that

$$\zeta(X,s) = \prod_{i=0}^{2\dim X} \det(1 - \mathrm{Frob}_q q^{-s} | H_c^i(X \times_F \overline{F}, \overline{\mathbb{Q}}_\ell)^{I_F})^{(-1)^{i+1}}, \qquad (1.1)$$

where H_c^i denotes the étale cohomology with compact support and the superscript I_F indicates elements fixed by the inertia subgroup. In general, $\zeta(X, s)$ is quite hard to compute if X has bad reduction.

In fact, we will compute the semisimple local factor, $\zeta^{\mathrm{ss}}(X, s)$, which reduces to (1.1), when I_F acts via a finite quotient (as it does when X has good reduction).

Definition 1.1. The semisimple local factor is given by,

$$\log \zeta^{\mathrm{ss}}(X, s) = \sum_{\alpha \geq 1} \sum_{i=0}^{2 \dim X} (-1)^i \operatorname{tr}^{\mathrm{ss}}(\mathrm{Frob}_q^{\alpha} \,|H_c^i(X \times_F \overline{F}, \overline{\mathbb{Q}}_\ell)) \frac{q^{-\alpha s}}{\alpha}. \tag{1.2}$$

The reader can already see how the calculation of the semisimple zeta function essentially requires the calculation of the semisimple trace. We apply the theory developed in sections 2 through 5 to the case where $X = \mathcal{M}_{\Gamma(p^r),M}$. However, we do not account for any contribution from the boundary in our calculation. This is concluded with Theorem 5.16.

2 Profinite Groups and Hecke Algebras

2.1 Some Representation theory

In order to both fix notation and to recall certain facts which will be useful to us, in this section we introduce some results from the theory of profinite groups and their representations. We refer the reader to [4] or [5] for a more complete treatment.

Canonically, a *locally profinite group* is a topological group G in which every open neighborhood of the identity contains an open compact subgroup. A group is called *profinite* if it is locally profinite and compact. Such a group comes endowed with a natural (Krull) topology. In fact,

$$G \xrightarrow{\sim} \varprojlim G/K\,, \tag{2.1}$$

where the limit is taken over open normal subgroups $K \subset G$ ordered by inclusion.

Let C be an algebraically closed field of characteristic zero. For our purposes, C will always be $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$. We will be working with representations of G over C-vector spaces. Precisely, a *representation* is a C-vector space V together with a group homomorphism

$$\pi : G \longrightarrow \mathrm{Aut}_C(V)\,. \tag{2.2}$$

In what follows we require representations which satisfy certain continuity and finiteness conditions, which we define presently. We say that a representation is *smooth* if for every $v \in V$ there exists $K_v \subset G$, an open compact subgroup, such that $\pi(k)v = v$ for every $k \in K_v$.

The smooth representations of G form an abelian category, which we denote by

$$\mathrm{Rep}_C(G). \tag{2.3}$$

If (π_1, V_1) and (π_2, V_2) are two objects of $\mathrm{Rep}_C(G)$, a morphism in this category is simply a linear transformation $f : V_1 \longrightarrow V_2$ which is compatible with π_1 and π_2. That is, $f(\pi_1(g)v) = \pi_2(g)f(v)$ for all $v \in V$ and $g \in G$.

Now let (π, V) be a smooth representation, and let $K \subset G$ denote an arbitrary compact open subgroup. We write V^K for the set of vectors fixed by $\pi(K)$. A representation is said to be *admissible* if V^K is finite dimensional for every compact open subgroup K.

We also say that (π, V) is *irreducible* if $V \neq \{0\}$ and V contains no nontrivial G-invariant subspace. In the above notation, if $\{0\} \subsetneq V' \subsetneq V$ is a proper subspace, then $V'^G \subsetneq V'$.

We will freely make use of several other types of representations, as needed. Notably, we will require $\mathrm{Ind}_H^G \pi$ the representation induced on G from a representation π of a subgroup H, as well as its cousin n-$\mathrm{Ind}_H^G \pi$ – the normalized induced representation [4, ch. 2.5]. Similarly we will refer to certain *supercuspidal* representations, the definition of which may be found in [4, ch. 4.1] or [3, p. 18]. Similarly, a good discussion of *tempered* representations may be found in [12, ch. 14].

We recall an important result from the theory of irreducible smooth representations:

Theorem 2.1 (Schur's Lemma)**.** If (π, V) is an irreducible smooth representation of G, then $\mathrm{End}_G(V) = C$.

Proof. See [4] or [5], for example. □

We will use an important consequence of this result. Let $Z \subset G$ be the center.

Corollary 2.2. There exists a character $\omega_\pi : Z \longrightarrow C^\times$ such that $\pi(z)v = \omega_\pi(z)v$ for all $v \in V$ and $z \in Z$.

Proof. Schur's lemma guarantees the existence of a homomorphism

$$\omega_\pi : Z \longrightarrow C^\times \tag{2.4}$$

such that π acts through ω_π. Let K be any compact open subgroup whose set of fixed vectors is nonzero. The morphism $\omega_\pi(g) = 1$ for every $g \in K \cap Z$. It follows that ω_π is a character of Z. □

Denoting the set of irreducible smooth C-representations of G by $\widehat{G}$, in the language of category theory, Corollary 2.2 says that there exists a map

$$\begin{aligned} \varphi : Z &\longrightarrow \mathrm{Hom}_{\mathrm{Set}}(\widehat{G}, C^{\times}) \\ z &\longmapsto \omega_{\cdot}(z) : \pi \mapsto \omega_{\pi}(z) . \end{aligned} \tag{2.5}$$

We would like to emphasize the following examples of the concepts developed so far, as they will be used throughout the rest of this paper.

Example 2.3. Let F be a nonarchimedian local field, with ring of integers $\mathfrak{o}_F$. Since $\mathfrak{o}_F$ is a discrete valuation ring, F inherits the usual p-adic absolute value. In fact, F is a topological field when it is given the metric topology induced by this absolute value. If we denote by $\mathfrak{m}$ the unique maximal ideal of $\mathfrak{o}_F$,

$$\mathfrak{o}_F \xrightarrow{\sim} \varprojlim \mathfrak{o}_F/\mathfrak{m}^n \tag{2.6}$$

and the additive group of F is locally profinite. The ideals $\mathfrak{m}^n$ form a system of open compact neighborhoods of 0.

Example 2.4. Similarly, $F^{\times}$ is locally profinite, and the unit groups $1+\mathfrak{m}^n$ give a compact open system of neighborhoods of 1.

Example 2.5. Most importantly for our purposes, the groups $G = \mathrm{GL}_n(F)$ are also profinite. This follows from the fact that G is an open subset of $M_{n\times n} \simeq F^{n^2}$, which is profinite given its natural (product) topology, inherited from F. Within $\mathrm{GL}_n(F)$, the groups $\mathrm{GL}_n(\mathfrak{o}_F)$ and $1 + \mathfrak{m}^n M_{n\times n}$ are open compact neighborhoods of 1.

2.2 Hecke Algebras

Here we recall the relationship between the smooth representations of a locally profinite group and certain modules over an algebra – the Hecke algebra. To begin, let us fix a Haar measure μ on a locally profinite group G.

Let $C_c^{\infty}(G)$ be the set of *smooth*, i.e. locally constant, complex-valued functions of compact support. The group G acts on $C_c^{\infty}(G)$ via both left and right translation. These provide smooth representations of G in the complex vector space $C_c^{\infty}(G)$. If f is an element of $C_c^{\infty}(G)$, then there exist K_1 and K_2, two open compact subgroups of G, such that

$$f(k_1 g) = f(g) = f(g k_2) \tag{2.7}$$

for every $g \in G$, $k_1 \in K_1$, and $k_2 \in K_2$.

We can compose $f_1, f_2 \in C_c^\infty(G)$ via the convolution operator:

$$f_1 * f_2(x) = \int_G f_1(g) f_2(g^{-1}x) d\mu(g). \tag{2.8}$$

We then have that $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$, which leads us to the following definition.

Definition 2.6. The *Hecke algebra* of a locally profinite group is the associative $\mathbb{C}$-algebra

$$\mathcal{H}(G) = (C_c^\infty, *). \tag{2.9}$$

For any compact open subgroup $K \subset G$, there is an associated idempotent element $e_K \in \mathcal{H}(G)$ defined by

$$e_K(x) = \begin{cases} \frac{1}{\mu(K)} & \text{if } x \in K \\ 0 & \text{otherwise .} \end{cases} \tag{2.10}$$

It is then not hard to show that the subalgebra

$$e_K * \mathcal{H}(G) * e_K = \{f \in \mathcal{H}(G) | f(k_1 g k_2) = f(g), g \in G, \text{ and } k_1, k_2 \in K\} \tag{2.11}$$

has e_K as an identity element for convolution. We set

$$\mathcal{H}(G, K) := e_K * \mathcal{H}(G) * e_K. \tag{2.12}$$

We then denote the center of $\mathcal{H}(G, K)$ by $\mathcal{Z}(G, K)$.

If we consider a smooth (complex) representation (π, V) of G, for any $f \in \mathcal{H}(G)$ we can define

$$\pi(f)v = \int_G f(g)\pi(f) v d\mu(g). \tag{2.13}$$

The map

$$(f, v) \longmapsto \pi(f)v \tag{2.14}$$

gives V the structure of an $\mathcal{H}(G)$-module. For a fixed compact open subgroup $K \subset G$, one can show (see, [5, p. 34]) that $\pi(e_K)$ is the projection $V \longrightarrow V^K$, and we have that V^K is an $\mathcal{H}(G, K)$-module.

2.3 The Bernstein Center

We now define $\widehat{\mathcal{H}}(G) := \varprojlim \mathcal{H}(G, K) \supset \mathcal{H}(G)$, where the limit is taken over all open compact subgroups $K \subset G$ ordered by inclusion. The transition functions for the limit are given by

$$\begin{aligned} e_K * H(G) * e_K &\longrightarrow e_{K'} * H(G) * e_{K'} \\ f &\longmapsto e_{K'} * f * e_{K'}. \end{aligned} \tag{2.15}$$

An element $\hat{h} \in \widehat{\mathcal{H}}(G)$ is then a system $(\hat{h}_{e_K})$ with $\hat{h}_{e_K} = e_{K'} * \hat{h}_{e_K} * e_{K'}$ whenever $K \subset K'$.

It is possible to think of $\widehat{\mathcal{H}}(G)$ as the space $\mathcal{D}(G)$ of distributions on G, with the condition that for all $T \in \mathcal{D}(G)$ and every open compact subgroup K, the distribution $T * e_K$ has compact support. By definition, such a distribution is a linear functional on $\mathcal{H}(G) = C_c^\infty(G)$. Explicitly, if we take $\psi \in \mathcal{H}(G)$, we have that $\psi \in \mathcal{H}(G, K_0)$, for some open compact subgroup K_0. Then, for any $\hat{h} \in \widehat{\mathcal{H}}(G)$, we have

$$\langle \hat{h}, \psi \rangle = \int_G \hat{h}_{e_{K_0}} * \psi(g) d\mu(g). \tag{2.16}$$

We wish to describe the center of $\widehat{\mathcal{H}}(G)$, which we call

$$\widehat{\mathcal{Z}}(G) = \varprojlim \mathcal{Z}(G, K). \tag{2.17}$$

The relationship (in fact, an equivalence of categories) between $\mathcal{H}(G)$-modules and smooth representations of G, mentioned at the end of Section 2.2, gives us a nice characterization of $\widehat{\mathcal{Z}}(G)$. It is the endomorphism ring of the identity functor in the category $\mathrm{Rep}_{\mathbb{C}}(G)$. For more details, see [3, ch. 1]. We proceed to give a more explicit description of the Bernstein center, $\widehat{\mathcal{Z}}(G)$, in a case of interest to us.

We begin with some general remarks about a reductive group G. Such a group possesses a maximal Zariski-connected solvable algebraic subgroup: this is the *Borel subgroup*, B. Any proper algebraic subgroup containing the conjugates of B is said to be a *parabolic subgroup*. Finally, every parabolic subgroup P contains a maximal reductive subgroup M, well defined up to conjugation, the *Levi subgroup*.

For the sake of clarity in the rest of this section, we set $G = \mathrm{GL}_n(F)$, where F is a nonarchimedian local field. In this case, the standard Borel subgroup consists of upper triangular matrices. The standard parabolic subgroup P can be described as the group of matrices of the form

$$\begin{pmatrix} B_1 & * & \dots & * \\ 0 & B_2 & & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & B_h \end{pmatrix}, \tag{2.18}$$

where the B_i are $n_i \times n_i$ nonsingular blocks such that $\sum_{i=0}^h n_i = n$ and $h > 1$. The Levi subgroup $M \subset P$ consists of matrices whose blocks above the diagonal are all zero. That is,

$$M \simeq \mathrm{GL}_{n_1} \times \dots \times GL_{n_h}. \tag{2.19}$$

There is an unramified character of M into the ring of global sections of the multiplicative group scheme. In the standard notation, we let $D = (\mathbb{G}_m)^k$. The character is then given by

$$\begin{aligned} \chi : M &\longrightarrow \mathcal{O}_D(D) \\ (B_1, \dots, B_h) &\longmapsto \textstyle\prod_{i=1}^h T^{v_p(\det(B_i))}, \end{aligned} \tag{2.20}$$

where we have used the identification $\mathcal{O}_D(D) \simeq \mathbb{C}[T_1, T_1^{-1}, \dots, T_k, T_k^{-1}]$. If we also fix a supercuspidal representation of σ of G, we can consider the induced representation

$$\text{n-}\operatorname{Ind}_M^G(\sigma\chi). \tag{2.21}$$

Somewhat abusively, we will use D to refer to the set of such representations. A proposition of Bernstein [3, Prop. 2.10] tells us that if we let $\operatorname{Rep}^a_{\mathbb{C}}(G)$ be the category of smooth admissible complex representations G, then

$$\operatorname{Rep}^a_{\mathbb{C}}(G) = \bigoplus_{(M,D)} \operatorname{Rep}^a_{\mathbb{C}}(G)(M, D), \tag{2.22}$$

where $\operatorname{Rep}^a_{\mathbb{C}}(G)(M, D)$ is the full subcategory consisting of representations which may be embedded in a direct sum (over parabolic subgroups with Levi subgroup M) of the form (2.21), where pairs (M, D) are considered up to conjugation.

Let $W(M, D) \subset \operatorname{Norm}_G(M)/M$ denote the subgroup of all n such that $\pi = n\pi n^{-1}$ for all representations $\pi \in D$. As in Corollary 2.2, any $z \in \widehat{\mathcal{Z}}(G)$ acts through a scalar on the representations of D. Furthermore, this action is invariant under $W(M, D)$ and defines a regular function on the scheme D. This yields:

Theorem 2.7 (Bernstein)**.** There exists an isomorphism

$$\widehat{\mathcal{Z}}(G) \simeq \coprod_{(M,D)} D/W(M, D) \tag{2.23}$$

where the disjoint union is taken over pairs (M, D) up to conjugation.

3 Counting Points on the Moduli Space of Elliptic Curves over a Finite Field

3.1 Elliptic Curves and their Moduli

In this section, we fix an integer $N \geq 3$. Let E/k be an elliptic curve over a field k. Formally, $E \longrightarrow \operatorname{Spec} k$ is a smooth proper 1-dimensional algebraic variety which is also a group object in the category of k-schemes. Supposing that N is not divisible by the characteristic p of k, it is well-known that $(\mathbb{Z}/N\mathbb{Z})^2 \simeq E[N]$. However, this isomorphism is not unique. The choice of such an isomorphism,

$$\phi : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N] \tag{3.1}$$

is called a *level-N-structure* for E.

In order to properly define the moduli space, we require slightly more general notion of elliptic curve. To broaden our definition, let T be an arbitrary k-scheme.

Definition 3.1. A *generalized elliptic curve* over T consists of a triple $(E, e, +)$, where $E \longrightarrow T$ is a flat proper T-scheme. We require that e be a section of S-smooth locus of E, i.e. $e \in E^{\mathrm{sm}}(T)$. Further, we demand that the operator $+ : E^{\mathrm{sm}} \times_T E \longrightarrow E$

- restricts to a commutative group scheme on E^{sm} with identity section e, and
- for every geometric point $\overline{x}$ such that $E_{\overline{x}}$ is singular, the action of E^{sm} on $E_{\overline{x}}$ (given by $+$) induces a rotation of the graph of irreducible components, $\Gamma(E_{\overline{x}})$.

(See [9, ch. II] for the reason why the last condition is required.)

Generalized elliptic curves also carry the notion of a level-structure:

Definition 3.2. Let N be an integer which is invertible on T. A *level-N-structure* of a generalized elliptic curve E/T is an isomorphism,

$$\phi : (\mathbb{Z}/N\mathbb{Z})^2_T \xrightarrow{\sim} E[N]_T, \tag{3.2}$$

where $(\mathbb{Z}/N\mathbb{Z})^2_T$ is the constant group scheme over T and $E[N]_T$ is the preimage of e under the endomorphism $[N] : E \longrightarrow E$.

Remark 3.3. If $E[N]_T$ is finite étale over T (in particular, if $E[N]_T \simeq (\mathbb{Z}/N\mathbb{Z})^2_T$) then N is invertible on T (see [15, p. 75]).

We wish to study families of (generalized) elliptic curves with level-N-structure, up to isomorphism. To do this, we consider the moduli space of such curves, $\mathcal{M}_N$. Although the explicit construction of $\mathcal{M}_N$ is quite technical (see [15], [9]), it can be thought of as a scheme over $\operatorname{Spec} \mathbb{Z}[1/N]$ whose points parametrize (isomorphism classes of) pairs (E, ϕ) of elliptic curves with level-N-structure.

More precisely, if we fix a base scheme $T/\operatorname{Spec} k$, then $\mathcal{M}_N$ represents the functor, from the category of T-schemes (on which N is invertible) to the category of sets [15, p. 104], given by

$$S \longmapsto \{(E_S/S, \phi), \text{curves with level-}N\text{-structure, up to isomorphism}\}, \tag{3.3}$$

where E_S is the pullback of E to a scheme over S. By its construction, the scheme $\mathcal{M}_N$ is a smooth affine curve over $\operatorname{Spec} \mathbb{Z}[1/N]$.

Thus we have that, for primes $p \nmid N$, the scheme $\mathcal{M}_N(\mathbb{F}_{p^\alpha})$ classifies elliptic curves with level-N-structure. Specifying an elliptic curve $E/\mathbb{F}_{p^\alpha}$, a technique due to Langlands, Kottwitz, and Rapoport [7] [18] gives an approach for counting the number of elliptic curves $E'/\mathbb{F}_{p^\alpha}$ which are isogenous to E, up to isomorphism. In the notation of [25], we count the points of

$$\{x \in \mathcal{M}_N(\mathbb{F}_{p^\alpha}) | E_x \text{ is isogenous to } E\} =: \mathcal{M}_N(\mathbb{F}_{p^\alpha})(E). \tag{3.4}$$

In order to state their main result, we must first introduce some concepts from the theories of étale homology and crystalline cohomology.

3.2 Étale Homology

Recall that for any elliptic curve $E/\mathbb{F}_{p^\alpha}$, the (ℓ-adic) Tate module of E is

$$T_\ell(E) = \varprojlim E[\ell^n], \tag{3.5}$$

where $E[\ell^{n+1}] \xrightarrow{[\ell]} E[\ell^n]$ is the multiplication-by-ℓ endomorphism. There is a natural $\mathbb{Z}_\ell$-module structure on $T_\ell(E)$ inherited from the $\mathbb{Z}/\ell^n\mathbb{Z}$-module structure of each $E[\ell^n]$. In fact, for $\ell \neq p$, $T_\ell(E)$ is a free $\mathbb{Z}_\ell$-module of rank 2. Let

$$T^p(E) = \prod_{\ell \neq p} T_\ell(E). \tag{3.6}$$

If we denote $\widehat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$, then $T^p(E)$ is a free $\widehat{\mathbb{Z}}^p$-module of rank 2 which is acted upon by $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^\alpha})$. In fact, T^p defines a functor:

$$\{\text{Elliptic curves over } \mathbb{F}_{p^\alpha}\} \xrightarrow{T^p} \{\text{Free } \mathbb{Z}^p\text{-modules (with Galois action)}\}. \tag{3.7}$$

Remark 3.4. It is also important to note that

$$T^p(E)/NT^p(E) \xrightarrow{\sim} E[N]. \tag{3.8}$$

Thus, a level-N-structure can be viewed as an isomorphism $(\mathbb{Z}/N\mathbb{Z})^2 \simeq T^p(E)/NT^p(E)$.

We now have all we need to introduce the first étale homology group of E:

Definition 3.5. Let $V^p = T^p(E) \otimes_{\mathbb{Z}} \mathbb{Q}$, and let

$$\mathbb{A}_f^p = \widehat{\mathbb{Z}}^p \otimes \mathbb{Q} = \prod_{\substack{\ell \neq p \\ \ell \neq \infty}} \mathbb{Q}_\ell \tag{3.9}$$

denote the ring of finite adèles whose p^{th} component is equal to 1. We have that V^p is then the first étale homology group of E (with coefficients in $\mathbb{A}_f^p$):

$$V^p = H_{1,\text{ét}}(E \times \overline{\mathbb{F}}_{p^\alpha}, \mathbb{A}_f^p) \tag{3.10}$$

The module $T^p(E)$ is a lattice in $V^p(E)$, which is stable under the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^\alpha})$.

3.3 Crystalline Cohomology

We start off by introducing some notation: Fix an algebraic closure of $\mathbb{Q}_p$ and let $\mathbb{Q}_{p^\alpha}$ denote the unique unramified extention of degree α of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. Also, let $\mathbb{Z}_{p^\alpha}$ be the ring of algebraic integers in $\mathbb{Q}_{p^\alpha}$. This notation is justified by the fact that $W(\mathbb{F}_{p^\alpha}) = \mathbb{Z}_{p^\alpha}$, where W denotes the ring of Witt vectors. We refer the reader to [10] for a nice construction of W.

We remark that the (arithmetic) Frobenius endomorphism,

$$\begin{aligned} \sigma : \mathbb{F}_{p^\alpha} &\longrightarrow \mathbb{F}_{p^\alpha} \\ x &\longmapsto x^{p^\alpha}, \end{aligned} \tag{3.11}$$

induces a map $\mathbb{Q}_{p^\alpha} \longrightarrow \mathbb{Q}_{p^\alpha}$, as $\mathrm{Gal}(\mathbb{Q}_{p^\alpha}/\mathbb{Q}_p) \simeq \mathrm{Gal}(\mathbb{F}_{p^\alpha}/\mathbb{F}_p)$. Thus, by abuse of notation, we can consider $\sigma : \mathbb{Z}_{p^\alpha} \longrightarrow \mathbb{Z}_{p^\alpha}$.

We now briefly recall the notion of a Dieudonné module, before using it to describe the crystalline cohomology of an elliptic curve. Let $W(\mathbb{F}_{p^\alpha})\{F, V\}$ denote the ring of noncommutative polynomials in variables F and V with coefficients in the ring of Witt vectors. We define the *Dieudonné ring* D, as $W(\mathbb{F}_{p^\alpha})\{F, V\}$ modulo the relations

- $Fw = w^\sigma F$,
- $Vw = w^{\sigma^{-1}} V$, and
- $FV = VF = p$

for $w \in W(\mathbb{F}_{p^\alpha})$.

There exists an anti-equivalence of categories, i.e. a contravariant functor,

$$\mathbb{D} : \{\text{Affine unipotent group schemes over } \operatorname{Spec} \mathbb{F}_{p^\alpha}\}^{\dagger} \longrightarrow \{D\text{-modules}\}. \tag{3.12}$$

For a construction of $\mathbb{D}$ and a proof of this anti-equivalence, we refer the reader to [10].

We now construct the Dieudonné module of our elliptic curve E. We begin with the inductive system $E[p^n] \hookrightarrow E[p^{n+1}]$, denoting its inductive limit by $E[p^\infty]$, and we define

$$T_p(E) := \mathbb{D}(E[p^\infty]) = \mathbb{D}(\varinjlim E[p^n]) = \varprojlim \mathbb{D}(E[p^n]) \tag{3.13}$$

As in the case of the Tate module, $T_p(E)$ is a free module of rank 2, this time over the ring $\mathbb{Z}_{p^\alpha} = W(\mathbb{F}_{p^\alpha})$. It inherits the σ-linear and σ^{-1}-linear actions of F and V, respectively. Indeed, T_p is a functor

$$\{\text{Elliptic curves over } \mathbb{F}_{p^\alpha}\} \xrightarrow{T_p} \{\text{Free } \mathbb{Z}_{p^\alpha}\text{-modules (with } F,\ V \text{ actions)}\}. \tag{3.14}$$

We are now ready to introduce the first crystalline cohomology group.

Definition 3.6. We have that $T_p(E) = H^1_{\text{cris}}(E)$, and $V_p(E) = T_p(E) \otimes_{\mathbb{Z}_{p^\alpha}} \mathbb{Q}_{p^\alpha}$ is an *isocrystal* (a $\mathbb{Q}_{p^\alpha}$-vectors space furnished with F and V).

As before, $T_p(E)$ is a lattice within $V_p(E)$ which is stable under the operations of F and $pF^{-1} = V$.

3.4 The Classical Langlands-Kottwitz Method

We now note that an isogeny of elliptic curves, $E' \xrightarrow{f} E$ gives isomorphisms,

$$\begin{aligned} f &: V^p(E') \xrightarrow{\sim} V^p(E) \text{ and} \\ f &: V_p(E') \xrightarrow{\sim} V_p(E) \end{aligned} \tag{3.15}$$

of $\mathbb{A}_f^p$-modules and $\mathbb{Q}_{p^\alpha}$-vector spaces respectively. Within these spaces, we have injections

$$\begin{aligned} f &: T^p(E') \hookrightarrow T^p(E) \text{ and} \\ f &: T_p(E') \hookrightarrow T_p(E). \end{aligned} \tag{3.16}$$

[†] An affine group scheme over a field is called *unipotent* if it contains no nontrivial multiplicative subgroup. These form a full subcategory of the category of affine group schemes over a given field. See [10, ch. II].

Thus, the isogeny defines a map $\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E) \longrightarrow V^p \times V_p$, given by

$$E' \longmapsto T^p(E') \times T_p(E') \subseteq V^p \times V_p\ . \tag{3.17}$$

Combining this with the facts recalled in Sections 3.2, 3.3, and in Remark 3.4, we are motivated to define the following sets:

$$Y^p = \{(T^p, \phi) |\, \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^\alpha})\text{-stable } \widehat{\mathbb{Z}}^p\text{-lattices in } V^p\} \tag{3.18}$$

where $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \simeq T^p \otimes \mathbb{Z}/NZ$, and

$$Y_p = \{T_p |\, F, pF^{-1}\text{-stable } \mathbb{Z}_{p^\alpha}\text{-lattices in } V_p\}. \tag{3.19}$$

We also denote by I the algebraic group $(\mathrm{End}(E/\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}} \mathbb{Q})^\times$.

We are now ready to state the correspondence of Langlands and Kottwitz. Unfortunately, the map (3.17) is not bijective. This is because the lattice $T^p(E') \times T_p(E')$ depends on the isogeny f. However, its equivalence class in the set $I(\mathbb{Q})\backslash(Y^p \times Y_p)$ does not.

Theorem 3.7. The map

$$\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E) \longrightarrow I(\mathbb{Q})\backslash(Y^p \times Y_p) \tag{3.20}$$

is a bijection.

Proof. Variants of this argument are given in [6], [18], and[25].

Injectivity:

Let (E_1, ϕ_1) and (E_2, ϕ_2) be elements of $\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E)$ which have the same image in $I(\mathbb{Q})\backslash(Y^p \times Y_p)$. The essential point of the proof is that any morphism $f \in \mathrm{Hom}(E_1, E_2) \otimes \mathbb{Q}$ induces a map of (co)homology groups, as in (3.15). It is sufficient to show that for any such f,

$$f \in \mathrm{Hom}(E_1, E_2) \iff \begin{cases} fT^p(E_1) \subset T^p(E_2) \text{ and} \\ fT_p(E_1) \subset T_p(E_2). \end{cases} \tag{3.21}$$

Let M be an integer such that $Mf \in \mathrm{Hom}(E_1, E_2)$, i.e.

$$\begin{tikzcd} E_1 \arrow[r, "Mf"] \arrow[d, "{[M]}"'] & E_2\ . \\ E_1 \arrow[ur, "f"] & \end{tikzcd} \tag{3.22}$$

We must show that $Mf = 0$ on $E_1[M]$. If $p \nmid M$, then

$$E_1[M] \simeq T^p(E_1)/MT^p(E_1). \tag{3.23}$$

Thus, $Mf(T^p(E_1)) \subset MT^p(E_2)$, and so

$$\begin{tikzcd}
T^p(E_1)/MT^p(E_1) \arrow[r, "Mf"] \arrow[d] & T^p(E_2)/MT^p(E_2) \arrow[d] \\
E_1[M] \arrow[r, "0"] & E_2[M].
\end{tikzcd} \tag{3.24}$$

For the case where $p|M$, by the Chinese remainder theorem we may assume that $M = p^r$. We use the fact that the functor $\mathbb{D}$ is an equivalence between the categories

$$\{\text{Finite flat } \mathbb{F}_{p^\alpha}\text{-group schemes}\} \xrightarrow{\sim} \{\mathbb{Z}_{p^\alpha}\text{-modules with } F \text{ and } V\}, \tag{3.25}$$

as well as that

$$\mathbb{D}(E_1[p^r]) \simeq \mathbb{D}(E_1)/p^r\mathbb{D}(E_1). \tag{3.26}$$

The above argument then applies (relying on the faithfulness of $\mathbb{D}$).
Surjectivity:

Let $((T^p, \phi), T_p)$ be an element of $Y^p \times Y_p$. Without loss of generality (applying elements of $I(\mathbb{Q})$ as necessary), we may assume that $T^p(E) \subset T^p$ and $T_p(E) \subset T_p$. We then rely on the fact that the finite flat subgroups of E correspond to subgroups of $T^p(E)\otimes\mathbb{A}_f^p/\widehat{\mathbb{Z}}^p$ which are $\mathrm{Gal}(\overline{\mathbb{F}}_p/F_{p^\alpha})$-invariant as well as to subgroups of $T_p(E) \otimes \mathbb{Q}_{p^\alpha}/\mathbb{Z}_{p^\alpha}$ which are F, V-invariant.We then have that

$$\begin{aligned} T^p(E) \otimes \mathbb{A}_f^p/\widehat{\mathbb{Z}}^p &\twoheadrightarrow T^p \otimes \mathbb{A}_f^p/\widehat{\mathbb{Z}}^p \text{ and} \\ T_p(E) \otimes \mathbb{Q}_{p^\alpha}/\mathbb{Z}_{p^\alpha} &\twoheadrightarrow T_p \otimes \mathbb{Q}_{p^\alpha}/\mathbb{Z}_{p^\alpha}\ , \end{aligned} \tag{3.27}$$

and the result follows. □

With this bijection in hand, we calculate the cardinality of $I(\mathbb{Q})\backslash(Y^p \times Y_p)$ in terms of certain orbital integrals. Ultimately, we find an expression for $|\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E)|$ in the language developed in Section 2.

3.5 Orbital Integrals and σ-Twisting

Let us fix an isomorphism of the étale homology, $V^p \simeq (\mathbb{A}_f^p)^2$ as $\mathbb{A}_f^p$-modules. The action of the Frobenius element Frob_{p^α}, when viewed as an element of $\mathrm{Aut}(T^p(E))$ (see the end of Section 3.2), induces an element $\gamma \in \mathrm{GL}_2(\mathbb{A}_f^p)$ which is well defined up to conjugation.

A theorem of Honda and Tate [6], which states that

$$\begin{aligned}\mathrm{End}(E)\otimes\mathbb{A}_f^p &= \mathrm{End}_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^\alpha}),\,\mathbb{A}_f^p\text{-lin}}(H_{1,\text{ét}}(E,\mathbb{A}_f^p)) && (3.28)\\ &= \mathrm{End}_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^\alpha}),\,\mathbb{A}_f^p\text{-lin}}(V^p), && (3.29)\end{aligned}$$

gives us the following description of the centralizer G_γ of γ:

$$I(\mathbb{A}_f^p)\simeq\{g\in\mathrm{GL}_2(\mathbb{A}_f^p)\,|\,g^{-1}\gamma g=\gamma\}=:G_\gamma(A_f^p). \tag{3.30}$$

Choosing an isomorphism of the crystalline cohomology, $V_p\simeq(\mathbb{Q}_{p^\alpha})^2$ as $\mathbb{Q}_{p^\alpha}$-vector spaces, allows us to explicitly describe the $\mathbb{Q}_{p^\alpha}$-linear endomorphism F, defined in Section 3.3. More precisely, after fixing a basis, we have two σ-linear bijections from $(\mathbb{Q}_{p^\alpha})^2$ to itself: One is F and the other is σ itself, extended as $\sigma(q_1,q_2)=(\sigma(q_1),\sigma(q_2))$. We pick $\delta\in\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$ such that $F=\delta\sigma$.

Claim 3.8. The element δ is well defined up to σ-conjugation.

Proof. Consider two different bases of V_p given by

$$V_p\xrightarrow{i_1}(\mathbb{Q}_{p^\alpha})^2 \text{ and } V_p\xrightarrow{i_2}(\mathbb{Q}_{p^\alpha})^2, \tag{3.31}$$

related by a change-of-basis matrix $g\in\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$: $i_1=g\circ i_2$. By definition, $i_1\circ F=\delta_1\sigma_1$ and $i_2\circ F=\delta_2\sigma_2$, where $\sigma_k=\sigma\circ i_k$. We proceed by comparing two different expressions for $i_2\circ F$. Dropping the circles,

$$i_2F=g^{-1}i_1F=g^{-1}(\delta_1\sigma_1)=\delta_2\sigma_2. \tag{3.32}$$

We also have that

$$\sigma_1=\sigma i_1=\sigma g i_2=g^\sigma\sigma i_2=g^\sigma\sigma_2. \tag{3.33}$$

Thus $g^{-1}\delta_1 g^\sigma\sigma_2=\delta_2\sigma_2$, and the conclusion follows. □

The proof of Claim 3.8 leads us to define the *twisted centralizer* as

$$G_{\delta\sigma}(\mathbb{Q}_{p^\alpha})=\{h\in\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})\,|\,h^{-1}\delta h^\sigma=\delta\} \tag{3.34}$$

Once again, Honda-Tate allows us to write,

$$\mathrm{End}(E)\otimes\mathbb{Q}_p=\mathrm{End}_{F,\,\mathbb{Q}_{p^\alpha}\text{-lin}}(H^1_{\mathrm{cris}}(E)\otimes\mathbb{Q}_{p^\alpha})=\mathrm{End}_{F,\,\mathbb{Q}_{p^\alpha}\text{-lin}}(V_p), \tag{3.35}$$

from which we get

$$I(\mathbb{Q}_p)\simeq G_{\delta\sigma}(\mathbb{Q}_{p^\alpha}). \tag{3.36}$$

We now require some additional notation. Let

$$X^p = \{(T^p, \phi) |\, \widehat{\mathbb{Z}}^p\text{-lattices in } V^p \text{ with } \phi : (\mathbb{Z}/N\mathbb{Z})^2 \simeq T^p \otimes \mathbb{Z}/NZ\} \supset Y^p, \tag{3.37}$$

and $X_p = \{T_p |\, \mathbb{Z}_{p^\alpha}\text{-lattices in } V_p\} \supset Y_p$.

An appropriate choice of basis for $T^p \subset Y^p \subset X^p$ and $T_p \subset Y_p \subset X_p$ allows us re-express X^p and X_p in group-theoretic language. Fix the following subgroups,

$$\begin{aligned} K^p &= \{g \in \mathrm{GL}_2(\widehat{Z}^p) |\, g \equiv 1 \mod N\} \text{ and} \\ K_{p^\alpha} &= \mathrm{GL}_2(\mathbb{Z}_{p^\alpha}). \end{aligned} \tag{3.38}$$

We may then write

$$X^p = \mathrm{GL}_2(\mathbb{A}_f^p)/K^p \text{ and also } X_p = \mathrm{GL}_2(\mathbb{Q}_{p^\alpha)}/K_p. \tag{3.39}$$

We relate this identification to the situation in Section 3.4 via the following proposition:

Proposition 3.9.

1. An element $g \in X^p$ defines a lattice in Y^p if and only if $g^{-1}\gamma g \in K^p$.
2. Likewise, $h \in X_p$ defines a lattice in Y_P if and only if $h^{-1}\delta h^\sigma \in K_{p^\alpha} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_{p^\alpha}$.

Proof.

1. By definition, Y^p is the set of fixed points of γ in X^p. An element $gK^p \in X^p$ is fixed by γ when $\gamma g K^p = gK^p$ i.e. when $g^{-1}\gamma g \in K^p$.
2. Similarly, Y_p is the set of elements of X_p which are invariant under F and V. Let T_p denote a lattice in X_p. Using the fact that $FV = p$, we can express the desired invariance as $pT_p \subsetneq FT_p \subsetneq T_p$, which we may write $pT_p \subsetneq h^{-1}\delta h^\sigma T_p \subsetneq T_p$. Let $g \in \mathrm{GL}_2(Q_{p^\alpha})$. It suffices to prove that

$$pT_p \subsetneq gT_p \subsetneq T_p \iff g \in K_{p^\alpha} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_{p^\alpha}. \tag{3.40}$$

The left side holds if and only if $T_p/gT_p \simeq \mathbb{F}_{p^\alpha}$. This is equivalent to being able to find a lattice basis (t_1, t_2) of T_p such that we can express gT_p as the span of $\left(\begin{smallmatrix} p \\ 0 \end{smallmatrix}\right) t_1$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) t_2$ with coefficients in $\mathbb{F}_{p^\alpha}$, i.e. the condition that

$$gT_p = (t_1, t_2) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{F}_{p^\alpha} \\ \mathbb{F}_{p^\alpha} \end{pmatrix}. \tag{3.41}$$

Of course, this is true if and only if we can find $k \in K_p$ such that $gT_p = k \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) T_p$, and $g \in K_{p^\alpha} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_{p^\alpha}$ as desired. We conclude by considering $g = h^{-1}\delta h^\sigma$.

□

We are now ready to reformulate our expression for $|\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E)|$. In anticipation of this result (Theorem 3.12), we prove the following lemmas: We make use of the shorthand,

$$\mathcal{G} = I(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f^p) \times \mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))/(K^p \times K_{p^\alpha}). \tag{3.42}$$

Lemma 3.10. Fix a Haar measure on $\mathrm{GL}_2(\mathbb{A}_f^p) \times \mathrm{GL}(\mathbb{Q}_{p^\alpha})$ which gives the compact open subgroup $K^p \times K_{p^\alpha}$ measure 1, and also fix a Haar measure on $I(\mathbb{Q})$ which gives points measure 1. Then, for $g \in \mathcal{G}$,

$$\mathrm{vol}_{I(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f^p)\times\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))}(I(\mathbb{Q})\backslash I(\mathbb{Q})g(K^p \times K_{p^\alpha})) = 1. \tag{3.43}$$

Proof. Using the definition of our normalized measure on $K^p \times K_{p^\alpha}$, the above volume is equal to

$$\frac{\mathrm{vol}_{\mathrm{GL}_2(\mathbb{A}_f^p)\times\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})}(g(K^p \times K_{p^\alpha})g^{-1})}{\mathrm{vol}_{I(\mathbb{Q})}(I(\mathbb{Q}) \cap g(K^p \times K_{p^\alpha})g^{-1}))} = \frac{1}{|I(\mathbb{Q}) \cap g(K^p \times K_{p^\alpha})g^{-1})|}. \tag{3.44}$$

We show that $|I(\mathbb{Q}) \cap g(K^p \times K_{p^\alpha})g^{-1})| = 1$. To do this, we use the fact (see [27, p. 100], among others) that $I(\mathbb{Q})$ is either a quadratic imaginary extention of $\mathbb{Q}$ or a quaternion algebra. In either case, every $h \in I(\mathbb{Q}) \cap g(K^p \times K_{p^\alpha})g^{-1}$ is contained in a maximal torus, $T = L^\times$ for some imaginary quadratic extention $L/\mathbb{Q}$. Since h is contained in the compact subgroup $g(K^p \times K_{p^\alpha})g^{-1}$, all of its eigenvalues are roots of unity. Thus h is one of the following $\{\pm 1, \pm i \pm \rho\}$, where ρ is a cube root of 1. Because h is conjugate to an element of K^p, its characteristic polynomial $P_h(x)$ satisfies

$$P_h(x) \equiv (x-1)^2 \equiv x^2 - 2x + 1 \mod N. \tag{3.45}$$

This rules out all cases except $N = 3$ and $h = \rho$. However, careful consideration of this case leads to a contradiction (see [18]), and we conclude. □

Lemma 3.10 is an easy, but critical step which allows us to calculate the cardinality of $I(\mathbb{Q})\backslash(Y^p \times Y_p)$. Let f^p be the characteristic function of $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$, and let ϕ_p be the characteristic function of $K_{p^\alpha} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_{p^\alpha} \subset \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$.

Lemma 3.11. The cardinality

$$|I(\mathbb{Q})\backslash(Y^p \times Y_p)| = \int_{\mathcal{G}} f^p(g^{-1}\gamma g)\phi_p(g^{-1}\delta g^\sigma)\frac{dg \times dh}{di}, \tag{3.46}$$

where $dg \times dh$ and di are the Haar measures on $\mathrm{GL}_2(\mathbb{A}_f^p) \times \mathrm{GL}(\mathbb{Q}_{p^\alpha})$ and $I(\mathbb{Q})$ respectively, as in Lemma 3.10.

Proof. We remark that by the expressions obtained in equation (3.39),

$$|I(\mathbb{Q})\backslash(X^p \times X_p)| = \sum_{\mathcal{G}} 1. \tag{3.47}$$

Hence, the left side of (3.46) is equal to

$$\sum_{\mathcal{G}} f^p(g^{-1}\gamma g)\phi_p(g^{-1}\delta g^\sigma). \tag{3.48}$$

We also have that the right hand side of 3.46 is equal to

$$\sum_{\mathcal{G}} \int_{K^p \times K_{p^\alpha}} f^p(g^{-1}\gamma g)\phi_p(g^{-1}\delta g^\sigma)\frac{dg \times dh}{di} = \tag{3.49}$$

$$\mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{Q})g(K^p \times K_{p^\alpha})) \int_{\mathcal{G}} f^p(g^{-1}\gamma g)\phi_p(g^{-1}\delta g^\sigma).$$

Thus we conclude by the calculation done in Lemma 3.10. □

We summarize our results in the theorem below. Let $f \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}_f^p))$, and define the *orbital integral*

$$\mathrm{O}_\gamma(f) = \int_{G_\gamma(\mathbb{A}_f^p)\backslash \mathrm{GL}_2(\mathbb{A}_f^p)} f(g^{-1}\gamma g)\frac{dg}{du}, \tag{3.50}$$

dg and du being the Haar measures on $\mathrm{GL}_2(\mathbb{A}_f^p)$ (as in Lemma 3.11) and on $G_\gamma(\mathbb{A}_f^p)$ respectively. Similarly, we define the *twisted orbital integral* as

$$\mathrm{TO}_{\delta\sigma}(\phi) = \int_{G_{\delta\sigma}(\mathbb{Q}_{p^\alpha})\backslash \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})} \phi(h^{-1}\delta h^\sigma)\frac{dh}{dt}, \tag{3.51}$$

for $\phi \in C_c^\infty(\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))$ and Haar measure dh on $\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$ and dt on $G_{\delta\sigma}(\mathbb{Q}_{p^\alpha})$.

Theorem 3.12.

$$|\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E)| = \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))\,\mathrm{O}_\gamma(f^p)\,\mathrm{TO}_{\delta\sigma}(\phi_p), \tag{3.52}$$

where the Haar measure on $\mathrm{GL}_2(\mathbb{A}_f^p) \times \mathrm{GL}(\mathbb{Q}_{p^\alpha}$ gives the compact open subgroup $K^p \times K_{p^\alpha}$ measure 1, and the Haar measure on $I(\mathbb{Q})$ gives points measure 1.

Proof. By Theorem 3.7, we already have a bijection with $I(\mathbb{Q})\backslash(Y^p \times Y_p)$. By the calculation done in Lemma 3.11, we have that

$$\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E)| = \int_{I(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f^p))\times\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})} f^p(g^{-1}\gamma g)\phi_p(g^{-1}\delta g^\sigma)\frac{dg \times dh}{di}$$

$$= V \times \int_{G_\gamma(\mathbb{A}_f^p)\backslash\,\mathrm{GL}_2(\mathbb{A}_f^p)} f^p(g^{-1}\gamma g)\frac{dg}{du} \times \int_{G_{\delta\sigma}(\mathbb{Q}_{p^\alpha})\backslash\,\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})} \phi_p(h^{-1}\delta h^\sigma)\frac{dh}{dt}.$$

The calculation of the volume term $V = \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))$ is done using di on $I(\mathbb{Q})$ and the measure induced on $I(\mathbb{A}_f)$ by $du \times dt$, making use of the fact that $I(\mathbb{A}_f) \simeq G_\gamma(\mathbb{A}_f^p) \times G_{\delta\sigma}(\mathbb{Q}_{p^\alpha})$. □

4 Some Algebraic Geometry

In this section we introduce some more advanced geometrical results in order to calculate the semisimple trace of Frobenius on the nearby cycle sheaves. Following the approach of Peter Scholze [25], we define these objects below.

4.1 Generalizing the Moduli Problem

We first introduce a slightly more sophisticated notion of moduli space which will allow us to consider elliptic curves with level-structure, even when the level is divisible by the characteristic of the base field. As in Section 3.1, we let p be the characteristic of base field F. We would like to consider $\mathcal{M}_N$ even when $N = p^r M$, for some $r > 0$ and $M \geq 3$. Fortunately, given an elliptic curve E/T, there is a natural generalization of the level-structure of Definition 3.2 which permits us to do exactly that:

Definition 4.1. An elliptic curve E/T is said to have *Drinfeld-level-p^r-structure*, sometimes denoted as a $\Gamma(p^r)$-structure, if there exist two sections $P, Q \in E(T)$ such that

$$E[p^r] = \sum_{(j,k)\in(\mathbb{Z}/p^r\mathbb{Z})^2} [jP + kQ] \tag{4.1}$$

as relative Cartier divisors on E.

(For a pertinent review of relative Cartier divisors, see [15, ch. 1].)

Since $E[N]$ is finite étale over T precisely when N is invertible (Remark 3.3), this definition is equivalent the ordinary notion of level-structure when T is a scheme over $\mathbb{Z}[1/p]$.

There is a corresponding moduli space $\mathcal{M}_{\Gamma(p^r),M}/\operatorname{Spec}\mathbb{Z}[1/M]$ that parametrizes (isomorphism classes of) elliptic curves which possess a level-M-structure for M prime to p and which also have a Drinfeld-level-p^r-structure for some $r > 0$. By its construction $\mathcal{M}_{\Gamma(p^r),M}$ represents a functor from the category of T-schemes to the category of sets given by

$$S \longmapsto \{(E_S/S, (P, Q), \phi)\}, \tag{4.2}$$

where (P, Q) specifies a $\Gamma(p^r)$-structure, ϕ gives the usual level-M-structure, and such triples are considered up to isomorphism (see [15, ch. 3] for details).

The scheme $\mathcal{M}_{\Gamma(p^r),M}$ is a regular affine curve over $\operatorname{Spec}\mathbb{Z}[1/M]$, and when viewed as a scheme over $\operatorname{Spec}\mathbb{Z}[1/pM]$, it is an étale covering space of $\mathcal{M}_M$ with Galois group $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$. The covering map is given by the finite forgetful morphism (in the category of $\operatorname{Spec}\mathbb{Z}[1/pM]$-schemes),

$$\pi_r : \mathcal{M}_{\Gamma(p^r),M} \longrightarrow \mathcal{M}_M\,. \tag{4.3}$$

It is an important fact that the special fiber of $\mathcal{M}_{\Gamma(p^r),M}$ admits a description as the union of certain regular divisors. For any subgroup $H \subset (\mathbb{Z}/p^r\mathbb{Z})^2$ of order p^r there is a reduced closed subscheme $\mathcal{M}^H_{\Gamma(p^r),M}$ of $\mathcal{M}_{\Gamma(p^r),M}$ which parametrizes elliptic curves with $\Gamma(p^r)$ structure such that

$$p^r[e] = \sum_{(j,k)\in H} [jP + kQ], \tag{4.4}$$

where $[e]$ is the divisor associated to the identity section of the elliptic curve. In fact, $\mathcal{M}^H_{\Gamma(p^r),M}$ is a regular divisor which is supported in the special fiber $\mathcal{M}_{\Gamma(p^r),M} \times_{\mathbb{Z}} \mathbb{F}_p$. The following theorem can be found in[8] and [25] and is proved in[15].

Theorem 4.2. Given any two subgroups $H, H' \subset (\mathbb{Z}/p^r\mathbb{Z})^2$ of order p^r, the regular divisors $\mathcal{M}^H_{\Gamma(p^r),M}$ and $\mathcal{M}^{H'}_{\Gamma(p^r),M}$ intersect at the points of $\mathcal{M}_{\Gamma(p^r),M} \times \mathbb{F}_p$ above supersingular elliptic curves, and

$$\mathcal{M}_{\Gamma(p^r),M} \times_{\mathbb{Z}} \mathbb{F}_p = \bigcup_{H\subset(\mathbb{Z}/p^r\mathbb{Z})^2} \mathcal{M}^H_{\Gamma(p^r),M}. \tag{4.5}$$

We proceed to define the semisimple trace and the complex of nearby cycles before calculating them in the case of interest to us.

4.2 Derived Categories and the Semisimple Trace

The semisimple trace was first introduced by Rapoport in [21]. In order to properly define it, we are obliged to work in certain derived categories. These objects are quite technical, and we merely describe their construction while fixing some notation. The reader is encouraged to consult [11] for an introduction and either [2] or [17] for a more detailed treatment.

Let G be a profinite group and consider the abelian category

$$\mathcal{R} = \mathrm{Rep}_{\mathbb{Z}/\ell^n\mathbb{Z}}(G) \tag{4.6}$$

of continuous representations of G in $\mathbb{Z}/\ell^n\mathbb{Z}$-modules of finite rank. The bounded derived category $D^b(\mathcal{R})$ is constructed as follows: First consider the category $\mathcal{K}^b(R)$ of bounded complexes of objects in $\mathcal{R}$, up to homotopy. We then localize this category, inverting the class of quasi-isomorphisms (maps which induce isomorphisms on cohomology). This is $D^b(\mathcal{R})$.

For our purposes, we will in fact need to work in the bounded "derived"[‡] category $D^b(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G))$ of the category of continuous finite-dimensional ℓ-adic representations of G.

For any finite extension E of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ and any object $K^\bullet \in \mathcal{R}$, we can define an object $K^\bullet \otimes E \in \mathrm{Rep}_E(G) =: \mathcal{R}(E)$. There is an analogous result for morphisms. These both extend to the derived category, and it is thus that we can consider a 2-limit of categories over all such E. Roughly, $D^b(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G))$ is defined by taking first a projective and then an inductive 2-limit of the kind just described:

$$D^b(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)) = \text{“}\varinjlim\text{”}\,\text{“}\varprojlim\text{”}\; D^b(\mathcal{R}(E)). \tag{4.7}$$

Although complicated to define, $D^b(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G))$ is quite well behaved: It is a triangulated category and possesses a standard t-structure. To review these concepts and the analogous construction of the derived category of $\overline{\mathbb{Q}}_\ell$-sheaves, consult [17, ch. 2 and appendix A].

Given a functor from $\mathcal{R}$ into another abelian category, the notion of derived functor in the derived category $D^b(\mathcal{R})$ naturally extends to a "derived" functor of $D^b(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G))$. We now specialize these notions and define the semisimple trace.

Let F be a local field whose residue field is $\mathbb{F}_q$ with $\ell \nmid q$. We consider the absolute Galois group $\Omega_F = \mathrm{Gal}(\overline{F}/F)$. Fix any geometric Frobenius element $\mathrm{Frob}_q \in \Omega_F$, i.e. the inverse of a lift of Frobenius to Ω_F [5, ch. 28]. There exists a function,

$$\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_q^r\,|\cdot) : D^b(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\Omega_F)) \longrightarrow \overline{\mathbb{Q}}_\ell, \tag{4.8}$$

[‡] This is not technically a derived category in the usual sense, though by abuse of terminology it is referred to as such.

which is additive in distinguished triangles.

To properly construct this map, we require Grothendieck's semistable reduction theorem [29]. Let I_F denote the inertia subgroup of Ω_F, defined by

$$1 \longrightarrow I_F \longrightarrow \Omega_F \longrightarrow \widehat{\mathbb{Z}} \longrightarrow 0\,. \tag{4.9}$$

Theorem 4.3 (Grothendieck)**.** If V is the $\mathbb{Q}_\ell$-vector space associated with a finite dimensional continuous ℓ-adic representation of Ω_F, then there exists a filtration

$$0 = V_0 \subset V_1 \subset \ldots \subset V_h = V \tag{4.10}$$

which is invariant under the action of $\Omega_F \longrightarrow \mathrm{GL}(V)$ and such that I_F acts via a finite quotient on V_{i+1}/V_i for all $0 \leq i \leq h-1$.

Proof. This result is given by Clozel [8] and Scholze [25], though it is essentially due to Grothendieck [29]. We outline the argument as follows: First, it suffices to show that the theorem holds for V_1.

Then, we remark that every maximal compact subgroup K of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ (i.e. $\mathrm{GL}(V)$ after fixing an isomorphism $V \simeq (\mathbb{Q}_\ell)^n$) is equivalent, up to conjugation, to $\mathrm{GL}_n(\mathbb{Z}_\ell)$. Fixing such a subgroup, there is a map $I_F \longrightarrow K$. In fact, we can take this map to be

$$I_F \longrightarrow K(1) := \{g \in K \,|\, g \equiv 1 \mod \ell\} \tag{4.11}$$

(by replacing I_F by a suitable subgroup of finite index, as needed).

This map factors through a certain surjection, $t_\ell : I_F \longrightarrow \mathbb{Z}_\ell$ after composition with a representation $\rho : \mathbb{Z}_\ell \longrightarrow K(1)$. These are compatible with the action of I_F:

$$i \cdot v = \rho(t_\ell(i)) \cdot v, \tag{4.12}$$

for $i \in I_F$. For any lift of Frobenius Frob_q, we have $t_\ell(\mathrm{Frob}_q^{-1}\, i\, \mathrm{Frob}_q) = q t_\ell(i)$.

Then, letting λ be a generator of the image of I_F in $\mathbb{Z}_\ell$, it follows that $\rho(\lambda)$ is conjugate to $\rho(\lambda)^q$. This implies that the eigenvalues of $\rho(\lambda)$ are all roots of unity, and the result follows. □

A filtration as in Theorem 4.3 is called *admissible*. Take V as above, along an admissible filtration $V_\bullet$.

Definition 4.4. The semisimple trace is given by,

$$\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_q^r\,|V) = \mathrm{tr}(\mathrm{Frob}_q^r\,|(\bigoplus_i V_{i+1}/V_i)^{I_F}) = \sum_i \mathrm{tr}(\mathrm{Frob}_q^r\,|(V_{i+1}/V_i)^{I_F}), \tag{4.13}$$

where the superscript I_F denotes the elements fixed by the inertia subgroup.

We check that the semisimple trace is in fact well defined.

Claim 4.5. The value of $\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_q^r|V)$ is independent of the chosen filtration.

Proof. Given two filtrations $U_\bullet$ and $V_\bullet$, let $W_\bullet$ be their common refinement. The claim then follows by the additivity of the trace: Consider, for example, the case when $V_\bullet \subset W_\bullet$. Then V_{i+1}/V_i decomposes as

$$V_{i+1}/V_i = \bigoplus_k W_{k+1}/W_k \tag{4.14}$$

for each i, and we are done. □

Since this definition of semisimple trace is independent of filtration and is additive in short exact sequences, it extends to the functor described in (4.8).

We use the semisimple trace precisely for its additivity. For example,

$$\mathrm{tr}(\mathrm{Frob}_q^r|V^{I_F}) \tag{4.15}$$

is not so well behaved. The relationship between the two traces can be described in terms of another "derived" functor,

$$R_{I_F} : D^b(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\Omega_F)) \longrightarrow D^b(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q))) \tag{4.16}$$

which "takes invariants under I_F." We then have that,

$$\mathrm{tr}(\mathrm{Frob}_q^r) \circ R_{I_F} = (1-q^r)\,\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_q^r). \tag{4.17}$$

Remark 4.6. The results of this section hold, mutatis mutandis, when we replace Ω_F by $\Omega_F \times H$, where H is a finite group. In Theorem 4.3, for example, we would require that the action of H on V commute with that of Ω_F.

4.3 $\overline{\mathbb{Q}}_\ell$-Sheaves and Nearby Cycles

To understand the complex of nearby cycle sheaves, the reader must first be familiar with the notion of a $\overline{\mathbb{Q}}_\ell$-sheaf. Canonically, we have

Definition 4.7. A *sheaf of* $\mathbb{Z}_\ell$*-modules* $\mathcal{F}$ is a system of constructable $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, $(\mathcal{F}_n, f_{n+1} : \mathcal{F}_{n+1} \longrightarrow \mathcal{F}_n)$ such that f_{n+1} induces an isomorphism,

$$\mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \xrightarrow{\simeq} \mathcal{F}_n \tag{4.18}$$

for all n.

We define a $\overline{\mathbb{Q}}_\ell$*-sheaf*, i.e. a sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces, on a scheme X to be a sheaf of $\mathbb{Z}_\ell$-modules $\mathcal{F}$ such that the étale cohomology of X, with coefficients in $\mathcal{F}$ is defined as

$$H^i_{\text{ét}}(X, \mathcal{F}) = (\varprojlim H^i_{\text{ét}}(X, \mathcal{F}_n)) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell. \tag{4.19}$$

(See [20, ch. 19].)

There is a "derived" category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ of $\overline{\mathbb{Q}}_\ell$-sheaves on X (obtained through a limiting process as in Section 4.2, see also [17].) Although we define certain structures for any $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, in what follows, it will often suffice to take $\mathcal{F}$ to be the constant sheaf $\overline{\mathbb{Q}}_\ell$.

We now introduce the complex of nearby cycles, as first developed in [29, I] and [30, XIII].

Suppose that F is a nonarchimedian local field with discrete valuation ring $\mathfrak{o}_F$ and residue field $\mathbb{F}_q$. We fix an algebraic closure $\overline{F}$ of F and consider the integral closure $\overline{\mathfrak{o}}_F$ of $\mathfrak{o}_F$ in $\overline{F}$.

Let S denote the scheme $\operatorname{Spec} \mathfrak{o}_F$ with its closed point s and generic point η. We also have $\overline{s}$ and $\overline{\eta}$, the geometric special and geometric generic points, respectively. Passing to spectra all around, we have,

$$\begin{tikzcd}
\operatorname{Spec}\overline{\mathbb{F}}_q \arrow[r] \arrow[dr] & \operatorname{Spec}\overline{\mathfrak{o}}_F \arrow[d] & \operatorname{Spec}\overline{F} . \arrow[l] \arrow[dl] \\
& S &
\end{tikzcd} \tag{4.20}$$

Take X to be an arbitrary S-scheme of finite-type, with its geometric special fiber $X_{\overline{s}}$ and geometric generic fiber $X_{\overline{\eta}}$. Also let $X_{\overline{\mathfrak{o}}_F}$ be the pullback of X to $\overline{\mathfrak{o}}_F$.

We also let $\iota : X_{\overline{s}} \longrightarrow X_{\overline{\mathfrak{o}}_F}$ and $j : X_{\overline{\eta}} \longrightarrow X_{\overline{\mathfrak{o}}_F}$ be the standard closed and open immersions of the geometric special and generic fibers of X/S. We have the diagram,

$$\begin{tikzcd}
X_{\overline{s}} \arrow[r, "\iota"] \arrow[d] & X_{\overline{\mathfrak{o}}_F} \arrow[d] & X_{\overline{\eta}} \arrow[l, "j"'] \arrow[d] \\
\operatorname{Spec}\overline{\mathbb{F}}_q \arrow[r, hook] & \operatorname{Spec}\overline{\mathfrak{o}}_F & \operatorname{Spec}\overline{F}. \arrow[l, hook']
\end{tikzcd} \tag{4.21}$$

Let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_\ell$-sheaf on X_η, and denote by $\mathcal{F}_{\overline{\eta}}$ the pullback of $\mathcal{F}$ to $X_{\overline{\eta}}$.

Definition 4.8. The *complex of nearby cycles* is defined as

$$R\psi\mathcal{F} = \iota^* Rj_* \mathcal{F}_{\overline{\eta}}. \tag{4.22}$$

It is a functor on the "derived" category of $\overline{\mathbb{Q}}_\ell$-sheaves on X_η :

$$R\psi : D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X \times_s \eta, \overline{\mathbb{Q}}_\ell), \tag{4.23}$$

where $D^b_c(X \times_s \eta, \overline{\mathbb{Q}}_\ell)$ is the category of $\mathcal{F} \in D^b_c(X_{\overline{s}}, \overline{\mathbb{Q}}_\ell$ together with a continuous action of $\mathrm{Gal}(\overline{F}/F)$ which is compatible with the action on $X_{\overline{s}}$.

The reader should consult [13] for a brief overview of nearby cycles, or [30, XIII] for a detailed treatment.

We specialize this notion with an explicit determination of the nearby cycles sheaves in a case of interest to us. This result will be critical in Section 5, as it will allow us calculate the semisimple trace of Frobenius over the nearby cycles of the modular curve.

We denote by F^∞ the maximal unramified extension of $F \supset \mathfrak{o}_F$ together with its ring of integers, $\mathfrak{o}_{F^\infty}$. If X/S is any scheme of finite type, we let X_{η^∞} and $X_{\mathfrak{o}_{F^\infty}}$ be the base changes of X to $\mathrm{Spec}\, F^\infty$ and $\mathrm{Spec}\, \mathfrak{o}_{F^\infty}$ respectively. We again have maps $\iota : X_{\overline{s}} \longrightarrow X_{\mathfrak{o}_{F^\infty}}$ and $j : X_{\eta^\infty} \longrightarrow X_{\mathfrak{o}_{F^\infty}}$. It follows that,

$$R_{I_F}(R\psi\mathcal{F}) = \iota^* R j_* \mathcal{F}_{\eta^\infty}, \tag{4.24}$$

as both sides are derivations of the same functor.

We make the additional hypotheses that X/S is a flat regular scheme of relative dimension 1 such that the special fiber X_s is the union of regular divisors (this should sound familiar, see Theorem 4.2). Fix x, an $\mathbb{F}_q$-point of X, and let W_1 be the $\overline{\mathbb{Q}}_\ell$-vector space spanned by the divisors passing through x. We define a function $W_1 \longrightarrow \overline{\mathbb{Q}}_\ell$ which sends the divisors passing through x to 1. Let $W_2 \subset W_1$ denote the kernel of this map. Under these assumptions, we have:

Theorem 4.9. There exist unique isomorphisms

$$(\iota^* R^k j_* \overline{\mathbb{Q}}_\ell)_x \simeq \begin{cases} \overline{\mathbb{Q}}_\ell & \text{if } k = 0 \\ W_1(-1) & \text{if } k = 1 \\ W_2(-1) & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.25}$$

Proof. This result relies on a method developed by Rapoport and Zink [22], as well as Thomason's purity theorem [28], and it is detailed in [25, p. 17]. □

Although Theorem 4.9 is the most important result of this section, in practice it is also important to understand how nearby cycles interact with cohomology. We conclude this section by recalling some classical theorems to this effect.

Theorem 4.10. If $X \longrightarrow S$ is proper, then there exists a $\mathrm{Gal}(\overline{F}/F)$-equivariant isomorphism

$$H^i_{\text{ét}}(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \simeq H^i_{\text{ét}}(X_{\overline{s}}, R\psi(\overline{\mathbb{Q}}_\ell)). \tag{4.26}$$

This result can be extended slightly, to schemes which are not proper.

Theorem 4.11. If $X \longrightarrow S$ is of finite-type but not proper, and there exists a compactification $X \hookrightarrow \overline{X}$, proper over S, such that the boundary $\overline{X} \backslash X$ is a relative normal crossings divisor, then there exists a $\mathrm{Gal}(\overline{F}/F)$-equivariant isomorphism of the cohomology groups with compact support:

$$H^i_c(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \simeq H^i_c(X_{\overline{s}}, R\psi(\overline{\mathbb{Q}}_\ell)). \tag{4.27}$$

To tie this in with our previous discussion, we consider the case $X = \mathcal{M}_{\Gamma(p^r),M}$. As its divisor at infinity is not étale over $\mathrm{Spec}(\mathbb{Z}[1/M])$, Theorem 4.11 is not directly applicable. However, an appropriate change of base does allow us to make a similar identification of cohomology groups, giving:

Theorem 4.12. There is a exists a $\mathrm{Gal}(\overline{F}/F)$-equivariant isomorphism,

$$H^i_c(\mathcal{M}_{\Gamma(p^r),M,\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \simeq H^i_c(\mathcal{M}_{\Gamma(p^r),M,\overline{s}}, R\psi(\overline{\mathbb{Q}}_\ell)). \tag{4.28}$$

Proof. This is [25, Thm. 7.11]. □

5 A Generalization of Langlands-Kottwitz After Scholze

In this section, we use the tools which we have developed in the rest of this paper to express the semisimple trace of Frobenius in terms of certain orbital integrals. Versions of these results were first given by Peter Scholze [25, Sec. 8 and 9].

5.1 Using Nearby Cycles to Calculate the Semisimple Trace of Frobenius

We now combine the expression of Theorem 4.9 along with the notions introduced in Sections 2.3, 3.5,4.1, and 4.3 in order to calculate the trace of Frobenius on the nearby cycles of the modular curve.

We begin by recalling some representation theory. We denote by B the Borel subgroup of GL_2. If we choose χ_1 and χ_2 characters of $(\mathbb{Z}/p^r\mathbb{Z})^\times$, we can construct a representation $\chi_1 \boxtimes \chi_2$ of $B(\mathbb{Z}/p^r\mathbb{Z})$ by setting

$$\chi_1 \boxtimes \chi_2 \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} = \chi_1(y_1)\chi_2(y_2). \tag{5.1}$$

(By definition, the kernel of this representation contains the unipotent radical of $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$.) Such a representation induces a representation of the entire general linear group, $\mathrm{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} \chi_1 \boxtimes \chi_2$. If we let 1 denote the trivial character (as well as the trivial representation), the *Steinberg representation* St is given by

$$\mathrm{St} = \ker(\mathrm{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes 1 \longrightarrow 1). \tag{5.2}$$

Now, recall that we have the covering map $\pi_r : \mathcal{M}_{\Gamma(p^r),M} \longrightarrow \mathcal{M}_M$. If we consider the constant sheaf $\overline{\mathbb{Q}}_\ell$ on the generic fiber $\mathcal{M}_{\Gamma(p^r),M,\eta}$, the pushforward gives us a sheaf

$$\mathcal{F}^r := \pi_{r,\eta,*}\overline{\mathbb{Q}}_\ell \tag{5.3}$$

on $\mathcal{M}_{M,\eta}$. Furthermore every point $x \in \mathcal{M}_M(\mathbb{F}_{p^\alpha})$ gives rise to an element $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$ by specifying a basis of the crystalline cohomology of the elliptic curve E_x. Using the notation of Section 3.5, we define $\mathrm{N}\,\delta := \delta\delta^\sigma \cdots \delta^{\sigma^{\alpha-1}}$.

Theorem 5.1. Let x be an $\mathbb{F}_{p^\alpha}$-point of $\mathcal{M}_M$ and g be an element of $\mathrm{GL}_2(\mathbb{Z}_p)$.

(i) If E_x is ordinary, and λ is the unique eigenvalue of $\mathrm{N}\,\delta$ with valuation 0, then

$$\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha}\, g | (R\psi\mathcal{F}^r)_x) = \mathrm{tr}(\mathrm{Frob}_{p^\alpha}\, g | V_r), \tag{5.4}$$

where

$$V_r \simeq \bigoplus_{\chi \in ((\mathbb{Z}/p^r\mathbb{Z})^\times)^\vee} \mathrm{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes \chi, \tag{5.5}$$

as a representation of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^\alpha}) \times \mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$. The Frobenius element Frob_{p^α} acts through the scalar $\frac{1}{\chi(\lambda)}$ on $\mathrm{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes \chi$.

(ii) If E_x is supersingular, then

$$\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha}\, g | (R\psi\mathcal{F}^r)_x) = 1 - \mathrm{tr}(g | \mathrm{St})p^\alpha. \tag{5.6}$$

Proof. We let $R\psi^X$ denote the nearby cycles over scheme X. We also freely use the notation of Section 4.3.

The finiteness of the covering map allows us to write,

$$R_{I_F}(R\psi^{\mathcal{M}_M}\pi_{r,\eta,*}\overline{\mathbb{Q}}_\ell) = \pi_{r,\overline{s},*}R_{I_F}(R\psi^{\mathcal{M}_{\Gamma(p^r),M}}\overline{\mathbb{Q}}_\ell) = \pi_{r,\overline{s},*}\iota^* Rj_*\overline{\mathbb{Q}}_\ell. \tag{5.7}$$

We then remark that Theorem 4.2 tells us that the modular curve satisfies the hypotheses of Theorem 4.9. We begin with the second assertion:

If E_x is supersingular, there is only one point in $\mathcal{M}_{\Gamma(p^r),M}$ above x. The irreducible components which intersect at the point above x are parametrized by $\mathbb{P}^1(\mathbb{Z}/p^r\mathbb{Z})$. As this parametrization is $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$-equivariant, we may immediately apply Theorem 4.9. We then have that,

$$W_1 \simeq \mathrm{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes 1. \tag{5.8}$$

By definition, $W_2 \simeq \mathrm{St}$, and the conclusion follows.

To prove the first assertion, when E_x is ordinary, we pick a point $x' \in \pi_r^{-1}(x)$. We once again apply Theorem 4.9 to obtain,

$$(\iota^* R^k j_*\overline{\mathbb{Q}}_\ell)_{x'} \simeq \begin{cases} \overline{\mathbb{Q}}_\ell & \text{if } k=0 \\ \overline{\mathbb{Q}}_\ell(-1) & \text{if } k=1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

To see what happens when we push this complex forward, we fix an isomorphism

$$E_x[p^\infty] \xrightarrow{\sim} \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p. \tag{5.10}$$

After also fixing a basis of the Dieudonné module $\mathbb{D}(E[p^\infty])$ and obtaining an element δ (see Section 3.5), this identification allows us to write,

$$\delta = \begin{pmatrix} p\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{5.11}$$

Using the definitions of Frob_{p^α} and N, we may also write

$$\mathrm{Frob}_{p^\alpha} = N\delta = \begin{pmatrix} p^\alpha N\lambda_1 & 0 \\ 0 & N\lambda_2 \end{pmatrix}. \tag{5.12}$$

We can already see that $\lambda = N\lambda_2$. This also implies that Frob_{p^α} acts by multiplication by $\frac{1}{\lambda}$.

The isomorphism in equation (5.10) involves identifying $E_x[p^r] \simeq \mu_{p^r} \times \mathbb{Z}/p^r\mathbb{Z}$ for each r. In turn, this gives rise to a family of surjections

$$(\mathbb{Z}/p^r\mathbb{Z})^2 \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z} \tag{5.13}$$

which parametrize the Drinfeld-level-p^r-structures. The Galois group of the covering, $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$, then acts on the Drinfeld structures by acting on the range of these maps.

Let $\mathfrak{S}$ denote the set of all surjections $(\mathbb{Z}/p^r\mathbb{Z})^2 \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z}$. Letting $V_r \simeq \overline{\mathbb{Q}}_\ell^{\mathfrak{S}}$, we combine what we have already said to get,

$$\pi_{r,\bar{s},*}\iota^* Rj_*\overline{\mathbb{Q}}_\ell \simeq \begin{cases} V_r & \text{if } k = 0 \\ V_r(-1) & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

The group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^\alpha}) \times \mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$ acts on the parameters of the Drinfeld structures and so also on V_r. The decomposition

$$V_r = \bigoplus_{\chi \in ((\mathbb{Z}/p^r\mathbb{Z})^\times)^\vee} \mathrm{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes \chi \tag{5.15}$$

follows after noticing that diagonal multiplication commutes with the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^\alpha}) \times \mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$. □

5.2 Base Change Lifts and Local Langlands

In order to reformulate the results of Section 5.1, we establish some facts about local base change, recalling the local Langlands correspondence in the process.

To begin, we examine the norm map $\mathrm{N}\,\delta$, introduced in the statement of Theorem 5.1. In general, that is for any $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$, we define

$$\mathrm{N}\,\delta := \delta\delta^\sigma \cdots \delta^{\sigma^{\alpha-1}}. \tag{5.16}$$

The importance of this norm comes from the relationship it establishes between centralizers. Although results that we will use hold in more general settings (see [23], for example), we will be content to work over GL_2.

Lemma 5.2. Let $\delta \in \mathrm{GL}_2(\mathbb{Z}_{p^\alpha}/p^r\mathbb{Z}_{p^\alpha})$.

(i) In the notation of Section 3.5, there is an equality,

$$|G_{\delta\sigma}(\mathbb{Z}/p^r\mathbb{Z})| = |G_{\mathrm{N}\,\delta}(\mathbb{Z}/p^r\mathbb{Z})|. \tag{5.17}$$

(ii) The map N gives a bijection between the σ-conjugacy classes of $\mathrm{GL}_2(\mathbb{Z}_{p^\alpha}/p^r\mathbb{Z}_{p^\alpha})$ and the conjugacy classes of $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$.

Proof. Surjectivity of the norm map can be established using a well-known result from group cohomology. Starting with any $\gamma \in \mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$, define two commutative groups,

$$\begin{aligned} Z_{\gamma,\mathbb{Q}_p} &= ((\mathbb{Z}/p^r\mathbb{Z})[\gamma])^\times \\ Z_{\gamma,\mathbb{Q}_{p^\alpha}} &= ((\mathbb{Z}_{p^\alpha}/p^r\mathbb{Z}_{p^\alpha})[\gamma])^\times. \end{aligned} \tag{5.18}$$

There is a map $d_1 : Z_{\gamma,\mathbb{Q}_{p^\alpha}} \longrightarrow Z_{\gamma,\mathbb{Q}_{p^\alpha}}$ which sends z to $zz^{-\sigma}$. Also, the norm map can be viewed as a function $d_2 : Z_{\gamma,\mathbb{Q}_{p^\alpha}} \longrightarrow Z_{\gamma,\mathbb{Q}_p}$.

To establish surjectivity, we would like to show that the complex

$$0 \longrightarrow Z_{\gamma,\mathbb{Q}_p} \hookrightarrow Z_{\gamma,\mathbb{Q}_{p^\alpha}} \xrightarrow{d_1} Z_{\gamma,\mathbb{Q}_{p^\alpha}} \xrightarrow{d_2} Z_{\gamma,\mathbb{Q}_p} \longrightarrow 0 \tag{5.19}$$

is exact. By definition, this is the same as showing that

$$\ker(d_2)/\operatorname{im}(d_1) =: H^1(\mathrm{Gal}(\mathbb{Q}_{p^\alpha}/\mathbb{Q}_p), Z_{\gamma,\mathbb{Q}_{p^\alpha}}) = 0. \tag{5.20}$$

Consider the filtration of $Z_{\gamma,\mathbb{Q}_{p^\alpha}}$ given by $Z^i := \{z \in Z_{\gamma,\mathbb{Q}_{p^\alpha}} \mid z \equiv 1 \mod p^i\}$, $i = 0, \dots, r$. We have that $Z^1/Z^0 \simeq (\mathbb{F}_{p^\alpha}[\gamma])^\times$, and for $i \geq 1$, $Z^i/Z^{i+1} \simeq \mathbb{F}_{p^\alpha}^4$. In either case, Hilbert's Theorem 90 [26] tells us that, $H^i(\mathrm{Gal}(\mathbb{Q}_{p^\alpha}/\mathbb{Q}_p), Z^i/Z^{i+1}) = 0$, and exactness follows.

We may now confidently find a $\delta \in Z_{\gamma,p^\alpha}$ such that $\gamma = \mathrm{N}\,\delta$. Take $h \in G_{\delta\sigma}(\mathbb{Z}/p^r\mathbb{Z})$. We remark that $h^{-1}\delta h^\sigma = \delta$ implies that $h^{-1}\,\mathrm{N}\,\delta h^\sigma = \mathrm{N}\,\delta$. Since h commutes with $\delta \in \mathbb{Z}_{p^\alpha}/p^r\mathbb{Z}_{p^\alpha}[\gamma]$, we conclude that $h = h^\sigma$, establishing that $G_{\delta\sigma}(\mathbb{Z}/p^r\mathbb{Z}) \subset G_\gamma(\mathbb{Z}/p^r\mathbb{Z})$. The inclusion $G_{\delta\sigma}(\mathbb{Z}/p^r\mathbb{Z}) \supset G_\gamma(\mathbb{Z}/p^r\mathbb{Z})$ is immediate. We have thus established part (i).

For part (ii), let $\gamma_1 \dots, \gamma_t$ represent the conjugacy classes of $\mathrm{GL}_2(\mathbb{Z}/p^r/Z)$, and choose δ_i such that $\gamma_i = \mathrm{N}\,\delta_i$, for $i = 1, \dots, t$. By part (i), we have an equality on the size of the σ-conjugacy classes,

$$\frac{|\,\mathrm{GL}_2(\mathbb{Z}_{p^\alpha}/p^r\mathbb{Z}_{p^\alpha})|}{|G_{\delta_i\sigma}(\mathbb{Z}/p^r\mathbb{Z})|} = \frac{|\,\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})|}{|G_{\gamma_i}(\mathbb{Z}/p^r\mathbb{Z})|} \tag{5.21}$$

Summing over classes gives,

$$\frac{|\,\mathrm{GL}_2(\mathbb{Z}_{p^\alpha}/p^r\mathbb{Z}_{p^\alpha})|}{|\,\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})|} \sum_i \frac{|\,\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})|}{|G_{\gamma_i}(\mathbb{Z}/p^r\mathbb{Z})|} = \frac{|\,\mathrm{GL}_2(\mathbb{Z}_{p^\alpha}/p^r\mathbb{Z}_{p^\alpha})|}{|\,\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})|} |\,\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})|. \tag{5.22}$$

Simplifying the last expression, we conclude that every element of $\mathrm{GL}_2(\mathbb{Z}_{p^\alpha}/p^r\mathbb{Z}_{p^\alpha})$ is σ-conjugate to one of the δ_i, $i = 1, \dots, t$, and we are done. □

An important, but straightforward consequence of Lemma 5.2 is the following: Let

$$\Gamma(p^r)_{\mathbb{Q}_p} = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) |\, g \equiv 1 \mod p^r\} \tag{5.23}$$
$$\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}} = \{g \in \mathrm{GL}_2(\mathbb{Z}_{p^\alpha}) |\, g \equiv 1 \mod p^r\} \tag{5.24}$$

be the standard principal congruence subgroups. Also, let $e_{\Gamma(p^r)_{\mathbb{Q}_p}}$ and $e_{\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}}$ respectively be their associated normalized idempotents (see Section 2.2).

Proposition 5.3. Let f be any locally integrable conjugation-invariant function on $\mathrm{GL}_2(\mathbb{Z}_p)$, and define the function ϕ on $\mathrm{GL}_2(\mathbb{Z}_{p^\alpha})$ by $\phi(\delta) = f(\mathrm{N}\,\delta)$. Then for every $\delta \in \mathbb{Z}_{p^\alpha}$,

$$e_{\Gamma(p^r)_{\mathbb{Q}_p}} * f(\mathrm{N}\,\delta) = e_{\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}} * \phi(\delta). \tag{5.25}$$

Proof. Without loss of generality, we assume that f is locally constant. The invariance of f under $\Gamma(p^r)_{\mathbb{Q}_p}$ immediately implies that ϕ is invariant under $\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}$. The function ϕ is then locally integrable. By our assumptions, and the definition of convolution, we have

$$e_{\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}} * \phi(\delta) = \int_{\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}} \phi(\delta) dg = \int_{\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}} f(\mathrm{N}(\delta)) dg. \tag{5.26}$$

Applying Lemma 5.2, (5.27) is equal to

$$\int_{\Gamma(p^r)_{\mathbb{Q}_p}} f(\mathrm{N}\,\delta) = e_{\Gamma(p^r)_{\mathbb{Q}_p}} * f(\mathrm{N}\,\delta). \tag{5.27}$$

. □

We are now almost ready to state an important theorem about base change. We first establish some vocabulary. Recall the notation for Hecke algebras found in Section 2.2. We will denote by

$$\mathrm{O}_\gamma(f) = \int_{G_\gamma(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)} f(g^{-1}\gamma g) d\mu(g) \tag{5.28}$$

and

$$\mathrm{TO}_{\delta\sigma}(\phi) = \int_{G_{\delta\sigma}(\mathbb{Q}_{p^\alpha})\backslash \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})} \phi(h^{-1}\delta h^\sigma) d\mu(h), \tag{5.29}$$

the usual orbital integrals, for any $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$, $f \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p))$, and any $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$, $\phi \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))$.

Definition 5.4. Two functions $f \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p))$ and $\phi \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))$ are said to have *matching orbital integrals* (or sometimes simply to be *associated*) if for every semisimple element $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$,

$$\mathrm{O}_\gamma(f) = \begin{cases} \pm \mathrm{TO}_{\delta\sigma}(\phi) & \text{if } \gamma \text{ is conjugate to } \mathrm{N}\,\delta \text{ for some } \delta. \\ 0 & \text{otherwise.} \end{cases} \tag{5.30}$$

We will also use the notion of a base-change lift, which we define presently. Let π be a tempered representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Definition 5.5. A representation Π of $\mathrm{GL}_2(\mathbb{Q}_{p^\alpha)}$ is said to be a *base-change lift* of π if

- Π is invariant under $\mathrm{Gal}(\mathbb{Q}_{p^\alpha}/\mathbb{Q}_p)$, and
- there exists an extention of Π to a representation of the semidirect product $\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}) \rtimes \mathrm{Gal}(\mathbb{Q}_{p^\alpha}/\mathbb{Q}_p)$ such that for every $g \in \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}_{p^\alpha}/\mathbb{Q}_p)$,

  $$\mathrm{tr}(\mathrm{N}\,g|\pi) = \mathrm{tr}((g,\sigma)|\Pi) \tag{5.31}$$

  whenever the conjugacy class of $\mathrm{N}\,g$ is regular semisimple[§].

A full discussion of base-change lifts can be found in [1]. We use them to state an important base-change theorem.

Recall that, after fixing a profinite group G, an element of the Bernstein center $z \in \widehat{\mathcal{Z}}(G)$ acts through a scalar on any irreducible smooth representation π of G (see Section 2.3). We call this scalar $c_{z,\pi}$.

Theorem 5.6. Let $f \in \widehat{\mathcal{Z}}(\mathrm{GL}_2(\mathbb{Q}_p))$ and $\phi \in \widehat{\mathcal{Z}}(\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))$ such that for any tempered irreducible smooth representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ with base change lift Π, $c_{f,\pi} = c_{\phi,\Pi}$.

(i) If $h \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p))$ and $h' \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))$ have matching orbital integrals, then $f * h$ and $\phi * h'$ are associated as well.

(ii) Furthermore, $e_{\Gamma(p^r)_{\mathbb{Q}_p}}$ and $e_{\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}}$ are associated.

Proof. This result is based on the work of Langlands [19] as well as on Kazhdan's density theorem [16]. It is due in its present form to Peter Scholze [25, Thm. 3.8]. Proof of the second statement also makes use of Proposition 5.3. □

[§] This means that every element of the conjugacy class is diagonalizable over the algebraic closure (*semisimple*) and that the eigenvalues of the elements of the class are distinct (*regular*).

Remark 5.7. Theorem 5.6 (ii) is the Fundamental Lemma in this setting.

Theorem 5.6 will be important in the reinterpretation of the results of Section 5.1. Before proceeding further, however, we also recall the local Langlands correspondence for GL_2. (See [5, ch. 7 and 8] for a good introduction.)

Implicitly, we fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. Given a p-adic ($p \neq \ell$) field F, there is a dense subgroup $\mathcal{W}_F$ of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ called the *Weil group.* Let $\mathcal{G}_2(F)$ denote the set of equivalence classes of 2-dimensional representations of the Weil group on which the action of Frobenius elements is semisimple. Also, let $\mathcal{A}_2(F)$ be set of equivalence classes of irreducible smooth admissible representations of $\mathrm{GL}_2(F)$.

Fact 5.8 (Local Langlands)**.** There is a unique bijection

$$\begin{tikzcd} \mathcal{A}_2(F) \arrow[r, "\sim"] & \mathcal{G}_2(F) \\ \pi \arrow[r, mapsto] & \sigma_\pi . \end{tikzcd} \tag{5.32}$$

5.3 Twisted Orbital Integrals and the Semisimple Trace of Frobenius

We turn once more to the setting of Section 5.1. We would like to consider the stalks of the nearby cycles sheaf at different levels. Recalling the notation of (5.3), we would like to work with $R\psi\mathcal{F}^r$ for different values r. With that in mind, set

$$(R\psi\mathcal{F}^\infty)_x = \varinjlim (R\psi\mathcal{F}^r)_x, \tag{5.33}$$

for a point $x \in \mathcal{M}_M(\mathbb{F}_{p^\alpha})$. Because the smooth action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on this sheaf commutes with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^\alpha})$, the semisimple trace of Frobenius extends in a natural way.

Fix $h \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Z}_p))$, and choose an r such that $h \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Z}_p), \Gamma(p^r)_{\mathbb{Q}_p})$. First, take invariants under the action of $\Gamma(p^r)_{\mathbb{Q}_p}$, then define

$$\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} h|(R\psi\mathcal{F}^\infty)_x) := \mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} h|(R\psi\mathcal{F}^r)_x). \tag{5.34}$$

Set $\gamma = \mathrm{N}\,\delta$, for the element δ associated to x. With this convention, the semisimple trace depends only on the value of γ.

Definition 5.9. Take $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$, $h \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Z}_p))$, and $\alpha \in \mathbb{N}$.

- For $v_p(\det \gamma) = \alpha$ and $v_p(\mathrm{tr}\, \gamma) \geq 0$:

- If $v_p(\operatorname{tr}\gamma) = 0$, then let λ_2 be unique eigenvalue of γ with $v_p(\lambda_2) = 0$, and define

$$c_\alpha(\gamma, h) = \sum_{\chi \in ((\mathbb{Z}/p^r\mathbb{Z})^\times)^\vee} \frac{\operatorname{tr}(h|\operatorname{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes \chi)}{\chi(\lambda_2)}. \tag{5.35}$$

- Also, for $v_p(\operatorname{tr}\gamma) \geq 1$, set

$$c_\alpha(\gamma, h) = \operatorname{tr}(h|1) - p^\alpha \operatorname{tr}(h|\operatorname{St}). \tag{5.36}$$

- Otherwise, set $c_\alpha(\gamma, h) = 0$.

With this shorthand, we are almost ready to state our main results for this section. First, however, we require a special function in the Bernstein center of $\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$.

Lemma 5.10. There exists a function $\widetilde{\varphi} \in \widehat{\mathcal{Z}}(\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))$ such that for every smooth irreducible representation Π of $\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$, $\widetilde{\varphi}$ acts on Π via the scalar

$$p^{\frac{1}{2}\alpha} \operatorname{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} |\sigma_\Pi). \tag{5.37}$$

As in Section 5.2, we use σ_Π to denote the representation of the Weil group associated with Π under the local Langlands correspondence.

Proof. This result makes use of the description of the Bernstein center given in Theorem 2.7, and it can be found in [25, sec. 9]. □

Let $\varphi_p = \widetilde{\varphi} * e_{\mathrm{GL}_2(\mathbb{Z}_{p^\alpha})}$. This function is an element of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}), \mathrm{GL}_2(\mathbb{Z}_{p^\alpha}))$.

Remark 5.11. We note that φ_p is, in fact, the normalized characteristic function,

$$\frac{1}{\operatorname{vol}(\mathrm{GL}_2(\mathbb{Z}_{p^\alpha}))} \times \mathbf{1}_{K_{p^\alpha}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)K_{p^\alpha}}. \tag{5.38}$$

Up to the normalization, it is equivalent to the function ϕ_p, of Section 3.5.

The following theorem unites the many of the concepts which we have covered so far. It is the last result which we will need in order to write down the semisimple trace of Frobenius using twisted orbital integrals.

Theorem 5.12. Let $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$, with $\mathrm{N}\,\delta = \gamma$ a semisimple element of $\mathrm{GL}_2(\mathbb{Q}_p)$. If $h \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Z}_p))$ and $h' \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Z}_{p^\alpha}))$ have matching orbital integrals, then

$$\mathrm{TO}_{\delta\sigma}(\widetilde{\varphi} * h') = \mathrm{TO}_{\delta\sigma}(\varphi_p) c_\alpha(\gamma, h). \tag{5.39}$$

Proof. First, let π be a tempered representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and Π its base-change lift. Also, choose an r such that h is $\Gamma(p^r)_{\mathbb{Q}_p}$-invariant and h' is $\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}$-biinvariant. Since h and h' are associated, from the definitions we have

$$\begin{aligned}\operatorname{tr}((\widetilde{\varphi} * h', \sigma)|\Pi) &= p^{\frac{1}{2}\alpha} \operatorname{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} |\sigma_\Pi) \operatorname{tr}((h', \sigma)|\Pi^{\Gamma(p^r)_{\mathbb{Q}_{p^\alpha}}}) &(5.40)\\ &= p^{\frac{1}{2}\alpha} \operatorname{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} |\sigma_\Pi) \operatorname{tr}(h|\pi^{\Gamma(p^r)_{\mathbb{Q}_p}}), &(5.41)\end{aligned}$$

where the exponents mean to take vectors fixed (see Section 2.1). At the same time, Theorem 5.6 tells us that $e_{\Gamma(1)_{\mathbb{Q}_p}}$ and $e_{\Gamma(1)_{\mathbb{Q}_{p^\alpha}}}$ have matching orbital integrals. Thus,

$$\operatorname{tr}((\varphi_p, \sigma)|\Pi) = p^{\frac{1}{2}\alpha} \operatorname{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} |\sigma_\Pi) \dim \pi^{\mathrm{GL}_2(\mathbb{Z}_p)}. \tag{5.42}$$

We now prove the theorem in a special case, from which the general result may be deduced (see Remark 5.13, below). To that effect, we make the assumption that δ is an element of the maximal torus T of $\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$ which consists of diagonal matrices. In particular, say

$$\delta = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \tag{5.43}$$

and that $\mathrm{N}\, t_1 \neq \mathrm{N}\, t_2$. Under these assumptions, we claim:

- If $v_p(t_1) = \alpha$ and $v_p(t_2) = 0$ (exchanging t_1 and t_2 if necessary), then

$$\begin{aligned}\mathrm{TO}_{\delta\sigma}(\widetilde{\varphi} * h') &= \frac{1}{\mathrm{vol}(T(\mathbb{Z}_p))} \sum_{\chi \in ((\mathbb{Z}/p^r\mathbb{Z})^\times)^\vee} \frac{\operatorname{tr}(h| \operatorname{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes \chi)}{\chi(t_2)} \\ &= \frac{1}{\mathrm{vol}(T(\mathbb{Z}_p))} c_\alpha(\gamma, h), &(5.44)\end{aligned}$$

  and

$$\mathrm{TO}_{\delta\sigma}(\varphi_p) = \frac{1}{\mathrm{vol}(T(\mathbb{Z}_p))}. \tag{5.45}$$

- Otherwise,

$$\mathrm{TO}_{\delta\sigma}(\widetilde{\varphi} * h') = \mathrm{TO}_{\delta\sigma}(\varphi_p) = 0. \tag{5.46}$$

Let $B \supset T$ be the standard Borel subgroup of GL_2 (see Section 2.3), and let χ be a unitary character of $T(\mathbb{Q}_p)$. The character χ induces an irreducible tempered representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, which we denote by

$$\pi_\chi := \text{n-}\operatorname{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi. \tag{5.47}$$

By the work of Langlands [19, ch. 7], there exists a character Θ_{π_χ} associated to the representation π_χ. This character is a locally integrable function on $\mathrm{GL}_2(\mathbb{Q}_p)$. More explicitly, if we identify $T(\mathbb{Q}_p)$ with $(\mathbb{Q}_p^\times)^2$, then for any element $t = (t_1, t_2) \in T(\mathbb{Q}_p)$ with distinct eigenvalues (i.e. regular),

$$\Theta_{\pi_\chi}(t) = \frac{\chi(t_1, t_2) + \chi(t_2, t_1)}{|\frac{(t_1-t_2)^2}{t_1t_2}|^{\frac{1}{2}}}. \tag{5.48}$$

(This is [19, Lemma. 7.2].)

If we fix a base-change lift Π_χ of π_χ, there is an analogous twisted character Θ_{Π_χ}, which is locally integrable on $\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$. By its definition, Θ_{Π_χ} satisfies,

$$\mathrm{tr}((f,\sigma)|\Pi_\chi) = \int_{\mathrm{GL}_2(\mathbb{Q}_{p^\alpha})} f(g)\Theta_{\Pi_\chi}(g)dg, \tag{5.49}$$

for any $f \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_{p^\alpha}))$.

For any $t \in T(\mathbb{Q}_p)$, set

$$\mathrm{TO}_t(f) := \begin{cases} \mathrm{TO}_{\tilde{t}\sigma}(f) & \text{if there exist } \tilde{t} \text{ with } t = \mathrm{N}\,\tilde{t} \\ 0 & \text{otherwise.} \end{cases} \tag{5.50}$$

Since all choices of $\tilde{t}$ are σ-conjugate, this definition is independent of $\tilde{t}$. With this convention, (5.49) is equivalent to

$$\frac{1}{2}\int_{T(\mathbb{Q}_p)} |\frac{(t_1-t_2)^2}{t_1t_2}|\,\mathrm{TO}_t(f)\frac{\chi(t_1,t_2)+\chi(t_2,t_1)}{|\frac{(t_1-t_2)^2}{t_1t_2}|^{\frac{1}{2}}}dt = \tag{5.51}$$
$$\int_{T(\mathbb{Q}_p)} |\frac{(t_1-t_2)^2}{t_1t_2}|^{\frac{1}{2}}\,\mathrm{TO}_t(f)\chi(t)dt.$$

Let $\widehat{T(\mathbb{Q}_p)}_u$ be set of unitary characters of $T(\mathbb{Q}_p)$. Using the Fourier inversion theorem, we get

$$\mathrm{TO}_t(f) = |\frac{(t_1-t_2)^2}{t_1t_2}|^{-\frac{1}{2}}\int_{\widehat{T(\mathbb{Q}_p)}_u} \frac{\mathrm{tr}((f,\sigma)|\Pi_\chi)}{\chi(t)}d\chi. \tag{5.52}$$

If we denote by $\widehat{T(\mathbb{Q}_p)}_u^\circ$ the set of unramified characters in $\widehat{T(\mathbb{Q}_p)}_u$, we must fix measures so that

$$\mathrm{vol}(\widehat{T(\mathbb{Q}_p)}_u^\circ) = \frac{1}{\mathrm{vol}(T(\mathbb{Z}_p))}. \tag{5.53}$$

Two straightforward calculations are all that remain: For $t = (t_1, t_2)$, if $v_p(t_1) = \alpha$ and $v_p(t_2) = 0$, then $\mathrm{TO}_t(\varphi_p) = \frac{1}{\mathrm{vol}(T(\mathbb{Z}_p))}$, and

$$\begin{aligned} \mathrm{TO}_t(\widetilde{\varphi} * h') &= p^{-\frac{1}{2}\alpha} \int_{\widehat{T(\mathbb{Q}_p)}_u} \frac{\mathrm{tr}((\widetilde{\varphi} * h', \sigma)|\Pi_\chi)}{\chi(t)} d\chi \qquad (5.54) \\ &= \frac{1}{\mathrm{vol}(T(\mathbb{Z}_p))} \sum_{\chi \in ((\mathbb{Z}/p^r\mathbb{Z})^\times)^\vee} \frac{\mathrm{tr}(h|\operatorname{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes \chi)}{\chi(t_2)}. \end{aligned}$$

In every other case, $\mathrm{TO}_{\cdot}(f)$ is a locally constant function on the regular elements of $T(\mathbb{Q}_p)$, and is thus the identity function there. It then follows that $\mathrm{TO}_t(\widetilde{\varphi} * h') = \mathrm{TO}_t(\varphi_p) = 0$ for all $t = (t_1, t_2)$, where $t_1 \neq t_2$. □

Remark 5.13. To prove Theorem 5.12 in the general case requires one to consider δ which are not σ-conjugate to elements of the form (5.43). For all such δ, the eigenvalues of $\mathrm{N}\,\delta$ have the same valuation. This case is treated in [25, Lemma 9.6].

Theorem 5.14. Let $x \in \mathcal{M}_M(\mathbb{F}_{p^\alpha})$ give rise to an element $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$. If $h \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Z}_p))$ and $h' \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Z}_{p^\alpha}))$ have matching orbital integrals, then

$$\mathrm{TO}_{\delta\sigma}(\widetilde{\varphi} * h') = \mathrm{TO}_{\delta\sigma}(\varphi_p)\,\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} h|(R\psi\mathcal{F}^\infty)_x). \qquad (5.55)$$

Proof. If we show that $\mathrm{N}\,\delta$ is semisimple, then the desired result will follow by Theorems 4.9 and 5.12. The element $\mathrm{N}\,\delta$ is an endomorphism of the crystalline cohomology of E_x. Given an elliptic curve $E'/\mathbb{F}_{p^\alpha}$, all we must prove is that an isogeny $E' \xrightarrow{f} E'$ induces a semisimple endomorphism of the crystalline cohomology.

For a contradiction, suppose there exist integers m and n such that the function $f' = mf - n$ is nilpotent on the crystalline cohomology, but is not identically zero. In that case, composing f' with its dual isogeny induces multiplication by a scalar on the crystalline cohomology. It follows that f' induces an invertible endomorphism on the rational crystalline cohomology, which is absurd. □

Theorem 5.15. If $x \in \mathcal{M}_M(\mathbb{F}_{p^\alpha})$ is associated to $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^\alpha})$, then

$$\mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} h|(R\psi\mathcal{F}^r)_x) = \frac{\mathrm{TO}_{\delta\sigma}(\widetilde{\varphi} * e_{\Gamma(p^r)_{\mathbb{Q}_{p^r}}})}{\mathrm{TO}_{\delta\sigma}(\varphi_p)} \qquad (5.56)$$

Proof. We apply Theorem 5.14, noting that by Theorem 5.6, we can take $h = e_{\Gamma(p^r)_{\mathbb{Q}_p}}$ and $h' = e_{\Gamma(p^r)_{\mathbb{Q}_{p^r}}}$. The fact that the right-hand side is well-defined is a consequence of Theorem 3.12, which tells us that $\mathrm{TO}_{\delta\sigma}(\varphi_p) \neq 0$, whenever $|\mathcal{M}_N(\mathbb{F}_{p^\alpha})(E)| \neq \varnothing$, the difference between ϕ_p and φ_p being inconsequential. □

We conclude with the computation of the semisimple factor of the Hasse-Weil zeta function for the modular curve. Once again, we do not take into account the contributions from infinity. As mentioned in Section 1, this is an important application of the calculations which we have just done.

Starting with Definition 1.1, there is a version of the Lefschetz trace formula [14, Prop. 10] may be combined with Theorem 4.12 in order to write

$$\log \zeta^{\mathrm{ss}}(\mathcal{M}_{\Gamma(p^r),M}), s) = \sum_{\alpha \geq 1} \sum_{x \in \mathcal{M}_M(\mathbb{F}_{p^\alpha})} \mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} \,|(R\psi \mathcal{F}^r)_x) \frac{p^{-\alpha s}}{\alpha}. \tag{5.57}$$

We can compute the sum over elements of $\mathcal{M}_M(\mathbb{F}_{p^\alpha})$ by summing over one isogeny class at a time. For a given elliptic curve $E/\mathbb{F}_{p^\alpha}$, we simply need to find

$$\sum_{x \in \mathcal{M}_M(\mathbb{F}_{p^\alpha})(E)} \mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} \,|(R\psi \mathcal{F}^r)_x). \tag{5.58}$$

Theorem 5.16.

$$\sum_{x \in \mathcal{M}_M(\mathbb{F}_{p^\alpha})(E)} \mathrm{tr}^{\mathrm{ss}}(\mathrm{Frob}_{p^\alpha} \,|(R\psi \mathcal{F}^r)_x) = \tag{5.59}$$
$$\mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f)) \, \mathrm{O}_\gamma(f^p) \, \mathrm{TO}_{\delta\sigma}(\widetilde{\varphi} * e_{\Gamma(p^r)_{\mathbb{Q}_{p^r}}}).$$

Proof. We have simply combined the final expressions of Theorems 5.15 and 3.12. □

References

[1] J. Arthur and L. Clozel. *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, volume 120 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, 1989.

[2] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100, pages 5–171. Asterisque, Soc. Math. France, 1982.

[3] J. Bernstein. Le "centre" de Bernstein. In P. Deligne, editor, *Représentations des groupes réductifs sur un corps local*, Travaux en Cours, pages 1–32. Hermann, Paris, 1984.

[4] Daniel Bump. *Automorphic Forms and Representations.* Cambridge University Press, 1997.

[5] C. Bushnell and G. Henniart. *The Local Langlands conjecture for GL(2)*, volume 335 of *A series of Comprehensive Studies in Mathematics.* Springer, Berlin, 2006.

[6] G. Chenevier. Notes de groupe de travail sur la fonction zeta des courbes modulaires, 2008–2009. Available at `http://www.math.polytechnique.fr/~chenevier/gt/gt.html`.

[7] L. Clozel. Nombre de points des variétés de Shimura sur un corps fini. In *Séminaire Bourbaki.*, volume 216, 121-149 (Exp. No.766). Société Mathématique de France, Astérisque. Paris, 1993.

[8] L. Clozel. Notes du cours: Fonctions zêta des courbes modulaires et correspondance de Langlands. Université de Paris-Sud Orsay, 2012.

[9] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In P. Deligne and W. Kuijk, editors, *Modular Functions of One Variable II*, volume 349 of *Lecture Notes in Mathematics.* Springer, Berlin, 1973.

[10] M. Demazure. *Lectures on p-divisible groups*, volume 302 of *Lecture Notes in Mathematics.* Springer, Berlin, 1972.

[11] E. Freitag and R. Kiehl. *Étale Cohomology and the Weil Conjecture.* Ergebnisse Der Mathematik Und Ihrer Grenzgebiete 3 Folge. Springer, Berin, 1988.

[12] D. Goldfeld and J. Hundley. *Automorphic Representations and L-Functions for the General Linear Group*, volume II. Cambridge University Press, Cambridge, 2011.

[13] T. Haines. Shimura varieties with parahoric level structure. In J. Arthur, D. Ellwood, and R. Kottwitz, editors, *Harmonic Analysis, the Trace Formula, and Shimura Varieties*, volume 4 of *Clay Mathematics Proceedings*, 2005.

[14] T. Haines and B.C. Ngô. Introduction to Shimura varieties with bad reduction of parahoric type. *Comp. Math.*, 133(02):117–150, 2002.

[15] N. Katz and B. Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, N.J., 1985.

[16] D. Kazhdan. Cuspidal geometry of p-adic groups. *J. Analyse Math.*, 47:1–36, 1986.

[17] R. Kiehl and R. Weissauer. *Weil conjectures, Perverse sheaves, and l-adic Fourier transform*, volume 42 of *A Series of Modern Surveys in Mathematics.* Springer, Berlin, 2001.

[18] R. Kottzitz. Isomorphism classes of elliptic curves within an isogeny class over a finite field. Notes from a class given at L'Université de Paris-Sud, Orsay.

[19] R. P. Langlands. *Base Change for GL(2)*, volume 96 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, 1980.

[20] J. S. Milne. Lectures on étale cohomology (v2.20), 2012. Available at `www.jmilne.org/math/`.

[21] M. Rapoport. On the bad reduction of Shimura varieties. In L. Clozel and J.S. Milne, editors, *Automorphic Forms, Shimura Varieties and L-functions II*. Academic Press, Inc., San Diego, 1990.

[22] M. Rapoport and T. Zink. über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindene Zyklen in ungleicher Charakteristik. *Inven. Math.*, 68(1):21–101, 1982.

[23] P. Scholze. The Langlands-Kottwitz approach for some simple Shimura varieties. *Preprint, Bonn*, 2010.

[24] P. Scholze. The local langlands correspondence for gln over p-adic fields. *Preprint, Bonn*, 2010.

[25] P. Scholze. The Langlands–Kottwitz approach for the modular curve. *Int. Math. Res. Not.*, 2011(15), 2011.

[26] J.-P. Serre. *Cohomologie Galoisienne*. Lecture Notes in Mathematics. Springer, Berlin, 5th edition, 1994.

[27] J. Silverman. *The Arithmetic of Elliptic Curves*. Springer, Berlin, 1986.

[28] R. Thomason. Absolute cohomological purity. *Bull. Soc. Mat. France*, 112(3):397–406, 1984.

[29] Groupes de monodromie en géométrie algébrique, 1972. Séminare de Géométrie Algébrique du Bois-Marie. 1967-1969 (SGA 7 I). Dirigé par A. Grothendieck avec la collaboration de M. Raynaud et D.S. Rim.

[30] Groupes de monodromie en géométrie algébrique, 1973. Séminare de Géométrie Algébrique du Bois-Marie. 1967-1969 (SGA 7 II). Dirigé par P. Deligne et N. Katz.